SUPERSYMMETRIC POLYNOMIALS AND THE CENTER OF THE WALLED BRAUER ALGEBRA

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Abstract. We study a commuting family of elements of the walled Brauer algebra $B_{r,s}(\delta)$, called the *Jucys-Murphy elements*, and show that the supersymmetric polynomials in these elements belong to the center of the walled Brauer algebra. When $B_{r,s}(\delta)$ is semisimple, we show that those supersymmetric polynomials generate the center. Under the same assumption, we define a maximal commutative subalgebra of $B_{r,s}(\delta)$, called the *Gelfand-Zetlin subalgebra*, and show that it is generated by the Jucys-Murphy elements. As an application, we construct a complete set of primitive orthogonal idempotents of $B_{r,s}(\delta)$, when it is semisimple. We also give an alternative proof of a part of the classification theorem of blocks of $B_{r,s}(\delta)$ in non-semisimple cases, which appeared in the work of Cox-De Visscher-Doty-Martin. Finally, we present an analogue of Jucys-Murpy elements for the quantized walled Brauer algebra $H_{r,s}(q,\rho)$ over $\mathbb{C}(q,\rho)$ and by taking the classical limit we show that the supersymmetric polynomials in these elements generates the center. It follows that H. Morton conjecture, which appeared in the study of the relation between the framed HOMFLY skein on the annulus and that on the rectangle with designated boundary points, holds if we extend the scalar from $\mathbb{Z}[q^{\pm 1},\rho^{\pm 1}]_{(q-q^{-1})}$ to $\mathbb{C}(q,\rho)$.

Introduction

The *Jucys-Murphy elements* of the group algebra $\mathbb{C}[\mathfrak{S}_r]$ of the symmetric group of r letters are given by

$$L_k := \sum_{j=1}^{k-1}(j,k) \quad (1 \le k \le r),$$

where (a,b) denotes the transposition exchanging a and b for $1 \le a,b \le r$. In particular, $L_1 = 0$ and L_k's are commuting to each other. These elements were introduced independently in [12, 24] and it was shown that the center of $\mathbb{C}[\mathfrak{S}_r]$ consists of all the symmetric polynomials in these elements ([12, 25]). This remarkable fact leads various interesting studies. For example, the ring homomorphism from the ring of symmetric polynomials to the center of $\mathbb{C}[\mathfrak{S}_r]$, which is called the *Jucys-Murphy specialization*, has been studied by many researchers. Let $f(x_1,\dots,x_r)$ be a symmetric polynomial. Since the evaluation $f(L_1,\cdots,L_r)$ belongs to

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the center of $\mathbb{C}[\mathfrak{S}_r]$, it can be written uniquely as a linear combination of the natural basis $\{C_\mu \mid \mu \text{ is a partition of } r\}$ of the center, where C_μ denotes the sum of all permutations with the same cycle type μ. That is, in the center of $\mathbb{C}[\mathfrak{S}_r]$, we have an equation

$$f(L_1, \dots, L_r) = \sum_\mu a_\mu^f C_\mu \quad (a_\mu^f \in \mathbb{C}).$$

The problem to calculate coefficients a_μ^f for various symmetric polynomials f is called the *class expansion problem.* For the elementary symmetric polynomial e_k and the power sum symmetric polynomial p_k, the class expansion problem has been completely solved in [12] and [17], respectively. For the monomial symmetric polynomial m_λ, a description of $a_\mu^{m_\lambda}$ for the partitions μ such that $\ell(\mu) = r - |\lambda|$ was given in [22], where $\ell(\mu)$ denotes the number of nonzero parts of μ and $|\lambda|$ denotes the sum of all the parts of λ. In [18], Lassalle solved the problem for a large family of symmetric polynomials including complete homogeneous symmetric polynomials h_k. Feray reproduced Lassalle's result in a different way ([9]). Another application of Jucys-Murphy elements is Okounkov and Vershik's beautiful approach to the representation theory of the symmetric groups ([28]). Observing simultaneous eigenspaces and eigenvalues of the Jucys-Murphy elements $L_1, \dots, L_r$ in an irreducible $\mathbb{C}[\mathfrak{S}_r]$-module, they gave a natural explanation about the appearance of Young diagrams and standard tableaux in the representation theory of symmetric groups.

The *walled Brauer algebra* $B_{r,s}(\delta)$ was introduced independently in [14] and [37] (See also [1]). When $\delta = n$, it was studied as the centralizer algebra of the action of general linear Lie algebra $\mathfrak{gl}(n)$ on the mixed tensor space. Moreover, when $\delta = m - n$, it is related with the centralizer algebra of the action of general linear Lie superalgebra $\mathfrak{gl}(m, n)$ on the mixed tensor superspace (See, for example, [2, 4, 34]). We call them the *mixed Schur-Weyl dualities.* As the mixed Schur-Weyl duality can be regarded as the generalization of the Schur-Weyl duality, one can consider the walled Brauer algebra as a natural generalization of the group algebra $\mathbb{C}[\mathfrak{S}_r]$ of the symmetric group $\mathfrak{S}_r$. Actually, in the diagrammatic description of the walled Brauer algebra $B_{r,s}(\delta)$, it is easily seen that a copy of the group algebra $\mathbb{C}[\mathfrak{S}_r]$ of the symmetric group $\mathfrak{S}_r$ is contained in it as a subalgebra. Thus it is natural to try to find a nice family of elements of $B_{r,s}(\delta)$ containing the Jucys-Murphy elements of $\mathbb{C}[\mathfrak{S}_r]$ and possessing similar properties with them.

In [2], Brundan and Stroppel defined a family of Jucys-Murphy elements $x_1^R, \dots, x_{r+s}^R$ of $B_{r,s}(\delta)$ and conjectured that the symmetric polynomials in $x_1^R, \dots, x_{r+s}^R$ generate the center of $B_{r,s}(\delta)$. In [32], Sartori and Stroppel worked in more general setting, which is called the *walled Brauer category.* The category includes the usual walled Brauer algebra $B_{r,s}(\delta)$ as an idempotent truncation. If our focus restricts to the case of $B_{r,s}(\delta)$, they defined the Jucys-Murphy elements $\xi_1, \dots, \xi_{r+s}$ and conjectured that the *doubly symmetric polynomials which satisfy the Q-cancellation property with respect to the r-th and $(r+1)$-th variables in* $\xi_1, \dots, \xi_{r+s}$ generate the center of $B_{r,s}(\delta)$. A doubly symmetric polynomial which satisfies the above Q-cancellation property is also called a *supersymmetric polynomial* in other literatures: it is a polynomial in $x_1, \dots, x_r, y_1, \dots, y_s$, symmetric in $x_1, \dots, x_r$ and in $y_1, \dots, y_s$ respectively, and the substitution $x_r = -y_1 = t$ yields a polynomial in $x_1, \dots, x_{r-1}, y_2, \dots, y_s$, which

is independent of t. Note also that in [29], Rui and Su introduced a family of elements of $B_{r,s}(\delta)$, called the *Jucys-Murphy-like elements*, in their study of affine walled Brauer algebras.

We are strongly motivated by the conjecture of [32]. In this paper, we define a family of Jucys-Murphy elements $L_1, \dots, L_{r+s}$ of $B_{r,s}(\delta)$, which can be regarded as a modification of the ones in [32], and show that supersymmetric polynomials in $L_1, \dots, L_{r+s}$ is central in $B_{r,s}(\delta)$. Here, we make use of some relations between generators of $B_{r,s}(\delta)$ and the Jucys-Murphy elements (Proposition 2.4). Another key ingredient is a theorem by Stembridge [36] saying that the ring of supersymmetric polynomials is generated by the *power sum supersymmetric polynomials*. We study the eigenvalues of the supersymmetric polynomials in Jucys-Murphy elements on the cell modules over $B_{r,s}(\delta)$. Our main theorem is that for the case when $B_{r,s}(\delta)$ is semisimple, which is the case except finitely many values δ, the supersymmetric polynomials in $L_1, \dots, L_{r+s}$ generate the center of $B_{r,s}(\delta)$ (Theorem 3.5). It follows by a modification of an argument, which was used by Li in [20] to produce a certain family of symmetric polynomials. As an application, we can mimic the Okounkov-Vershik's approach to the representation theory of the symmetric groups: when $B_{r,s}(\delta)$ is semisimple, we define the *Gelfand-Zetlin subalgebra* of $B_{r,s}(\delta)$ as the subalgebra generated by centers of certain naturally chosen subalgebras, which are isomorphic to walled Brauer algebras of lower ranks, and show that it is generated by the Jucys-Murphy elements $L_1, \dots, L_{r+s}$. Note that the Gelfand-Zetlin subalgebra is a maximal commutative subalgebra of $B_{r,s}(\delta)$, since the branching graph of $B_{r,s}(\delta)$ with respect to our choice of the family of subalgebras is multiplicity-free. A complete set of primitive orthogonal idempotents of $B_{r,s}(\delta)$ can now be constructed easily. In addition, by observing a connection between the eigenvalues of the supersymmetric polynomials in Jucys-Murphy elements and the conditions in the characterization of blocks of $B_{r,s}(\delta)$ in non-semisimple cases, we can recover a part of the classification theorem of blocks of $B_{r,s}(\delta)$ appeared in [3] (Proposition 5.3). This strengthens our belief that the center of $B_{r,s}(\delta)$ is generated by the supersymmetric polynomials in Jucys-Murphy elements, even when $B_{r,s}(\delta)$ is not semisimple (Conjecture 5.4). Lastly, we consider the case of the *quantized walled Brauer algebras*. A family of one parameter deformation of $B_{r,s}(N)$ ($N \in \mathbb{Z}_{\geq 0}$) has been appeared in [16] and a two parameter version has been introduced in [15] and [19]. Surprisingly enough, in his study of a connection between the *framed HOMFLY skein module on the annulus* and the one on the rectangle with designated input and output boundary points ([27]), Hugh Morton conjectured that the center of the quantized walled Brauer algebra over Λ, a certain localization of the ring of Laurent polynomials of two variables, is generated by the supersymmetric polynomials in some commuting elements, so called *Murphy operators*, which is a generalization of the ones in [7]. It turns out that Morton's elements are natural deformation of our Jucys-Murphy elements (See Definition 6.2 and Remark 6.7(2)). By taking a suitable limit sending q to 1, we can use our main theorem to show that the supersymmetric polynomials in those elements generate the center of the quantized walled Brauer algebra over $\mathbb{C}(q, \rho)$ and hence Morton's conjecture holds provided the base ring is extended from Λ to $\mathbb{C}(q, \rho)$.

This paper is organized as follows: In Section 1, we briefly recall the definition of walled Brauer algebras $B_{r,s}(\delta)$ and their cell modules. In Section 2, we introduce the Jucys-Murphy elements $L_1, \dots, L_{r+s}$ of $B_{r,s}(\delta)$ and prove several relations between generators of $B_{r,s}(\delta)$ and

the Jucys-Murphy elements. Using these relations, we show that the supersymmetric polynomials in $L_1, \dots, L_{r+s}$ belong to the center of $B_{r,s}(\delta)$. In Section 3, we calculate the eigenvalues of supersymmetric polynomials in the Jucys-Murphy elements on cell modules. Based on this calculation, we prove that when $B_{r,s}(\delta)$ is semisimple, the supersymmetric polynomials in $L_1, \dots, L_{r+s}$ generates the center of $B_{r,s}(\delta)$. In Section 4, we define the Gelfand-Zetlin subalgebra of $B_{r,s}(\delta)$ and show that it is generated by $L_1, \dots, L_{r+s}$ when $B_{r,s}(\delta)$ is semisimple. In Section 5, we give an alternative proof of a part of the classification theorem of blocks of $B_{r,s}(\delta)$ given in [3]. Lastly, we present an analogue of Jucys-Murpy elements for the quantized walled Brauer algebra $H_{r,s}(q, \rho)$ and we show that the supersymmetric polynomials in these elements generate the center.

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1. Walled Brauer algebras and their Representations

In this section, we will recall the walled Brauer algebras and their cell modules. We mainly follow the exposition in the papers [2, 3].

1.1. Walled Brauer algebras.

Let r and s be nonnegative integers. An (r, s)-*walled Brauer diagram* is a graph consisting of two rows with $r+s$ vertices in each row such that the following conditions hold:

(1) Each vertex is connected by a strand to exactly one other vertex.
(2) There is a vertical wall which separates the first r vertices from the last s vertices in each row.
(3) A *vertical strand* connects a vertex on the top row with one on the bottom row, and it cannot cross the wall. A *horizontal strand* connects vertices in the same row, and it must cross the wall.

Note that the vertical strands are called the *propagating lines* and the horizontal strands on the top row (respectively, on the bottom row) are called the *northern arcs* (respectively, *southern arcs*) in [3].

Let δ be a complex number and let us denote $B_{r,s}(\delta)$ the $\mathbb{C}$-vector space spanned by the basis consisting of all the (r, s)-walled Brauer diagrams. The dimension of $B_{r,s}(\delta)$ equals to $(r+s)!$ (see, for example [2, (2.2)]). We define a multiplication of (r, s)-walled Brauer diagrams as follows: For (r, s)-walled Brauer diagrams d_1, d_2, we put d_1 under d_2 and identify the top vertices of d_1 with the bottom vertices of d_2. We remove the loops in the middle row, if there exist. Then thus obtained diagram, denoted by $d_1 * d_2$, is again an (r, s)-walled Brauer diagram. We define the multiplication of d_1 by d_2

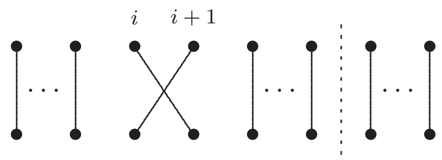
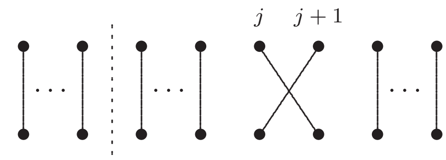
$$d_1 d_2 := \delta^n \, d_1 * d_2 \in B_{r,s}(\delta),$$

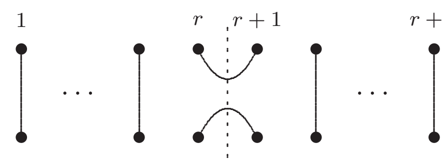
where n denotes the number of loops we removed in the middle row. Extending this multiplication by linearity, we obtain a multiplication on $B_{r,s}(\delta)$, which can be easily seen to be associative. We call thus obtained $\mathbb{C}$-algebra the *walled Brauer algebra*.

For each (r,s)-walled Brauer diagram, we number the vertices in each row of it by $1, 2, \ldots, r+s$ in the order from left to right. Then we have the following set of generators

$$\{s_i \mid 1 \leq i \leq r-1\} \cup \{s_j \mid r+1 \leq j \leq r+s-1\} \cup \{e_{r,r+1}\}$$

of the algebra $B_{r,s}(\delta)$ given by

$s_i :=$  , $s_j :=$  ,

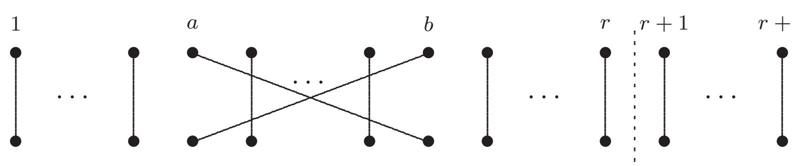
$e_{r,r+1} :=$  .

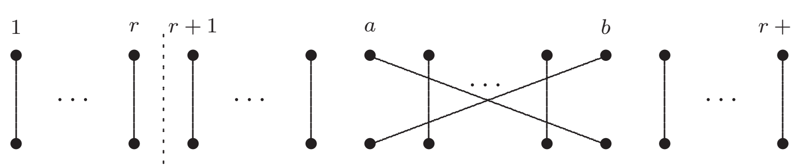
Let $0 \leq t, t' \leq \min(r,s)$. The subalgebra of $B_{r,s}(\delta)$ generated by $s_1, \ldots, s_{r-t-1}$ and $s_{r+t'+1}, \ldots, s_{r+s-1}$ can be identified with the group algebra $\mathbb{C}[\mathfrak{S}_{r-t} \times \mathfrak{S}_{s-t'}] \simeq \mathbb{C}[\mathfrak{S}_{r-t}] \otimes \mathbb{C}[\mathfrak{S}_{s-t'}]$, where $\mathfrak{S}_k$ denotes the symmetric group of k letters. We will use this identification for the rest of the paper. Let us define

$$\begin{aligned}
&(a,b) = (b,a) := s_{b-1} \cdots s_{a+1} s_a s_{a+1} \cdots s_{b-1}, \text{ for } 1 \leq a < b \leq r \text{ or } r+1 \leq a < b \leq r+s,\\
&e_{j,k} := (s_{k-1} \cdots s_{r+2} s_{r+1})(s_j \cdots s_{r-2} s_{r-1}) e_{r,r+1} (s_{r-1} s_{r-2} \cdots s_j)(s_{r+1} s_{r+2} \cdots s_{k-1})\\
&\hspace{20em} \text{for } 1 \leq j \leq r,\ r+1 \leq k \leq r+s.
\end{aligned}$$

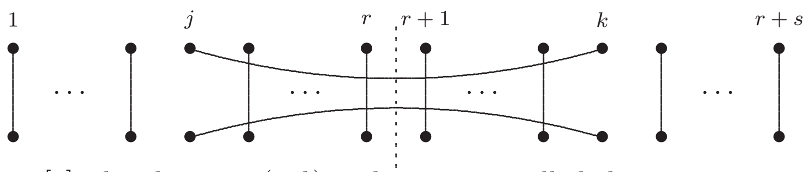
Indeed, the above elements have simple forms as diagrams:

$(a,b) = (b,a) =$

 if $1 \leq a < b \leq r$,

 if $r+1 \leq a < b \leq r+s$,

and

$e_{j,k} =$  .

Note that in [2], the elements (a, b) and $-e_{j,k}$ are called the *transpositions in* $B_{r,s}(\delta)$.

There is a criterion of the semisimplicity for $B_{r,s}(\delta)$:

Proposition 1.1. ([3, Theorem 6.3]) *The walled Brauer algebra $B_{r,s}(\delta)$ is semisimple if and only if one of the followings holds:*
(1) $r = 0$ *or* $s = 0$,
(2) $\delta \notin \mathbb{Z}$,
(3) $|\delta| > r + s - 2$,
(4) $\delta = 0$, *and* $(r, s) \in \{(1, 2), (1, 3), (2, 1), (3, 1)\}$.

Hence the algebra $B_{r,s}(\delta)$ is semisimple except for finitely many values $\delta \in \mathbb{C}$ for a fixed pair (r, s). Note that an analogue of the above for the quantized walled Brauer algebra is proved in [30, Theorem 6.10].

1.2. **Cell modules.** A *partition* is a weakly decreasing sequence of nonnegative integers with finitely many nonzero entries. For a partition $\mu = (\mu_1, \mu_2, \cdots)$, set $|\mu| := \sum_{i\geq 1} \mu_i$ and set $\ell(\mu) := |\{i \geq 1 \mid \mu_i \neq 0\}|$. Sometimes, we identify a partition $\mu = (\mu_1, \mu_2, \cdots)$ with the Young diagram whose boxes are arranged in left-justified rows with the lengths $\mu_1, \mu_2, \cdots$, and denote it by $[\mu]$.

Let us denote Λ the set of pair of partitions. An element in Λ is called a *bipartition*. Set

$$\Lambda^t_{r,s} := \left\{(\lambda^L, \lambda^R) \in \Lambda \mid |\lambda^L| = r - t,\ |\lambda^R| = s - t\right\}, \quad \Lambda_{r,s} := \bigsqcup_{t=0}^{\min(r,s)} \Lambda^t_{r,s},$$

$$\dot{\Lambda}_{r,s} = \begin{cases} \Lambda_{r,s} & \text{if } \delta \neq 0 \text{ or } r \neq s \text{ or } r = s = 0, \\ \Lambda_{r,s} - \{(\emptyset, \emptyset)\} & \text{if } \delta = 0 \text{ and } r = s \neq 0. \end{cases}$$

Define $J^k_{r,s}$ to be the subspace of $B_{r,s}(\delta)$ spanned by the (r, s)-walled Brauer diagrams with at least k horizontal strands at the top and at the bottom. Then $J^k_{r,s}$ is a two-sided ideal of $B_{r,s}(\delta)$ and $J^{k'}_{r,s} \subset J^k_{r,s}$ for $k' \geq k$.

Let $\lambda = (\lambda^L, \lambda^R)$ be an element in $\Lambda^t_{r,s}$ for some $0 \leq t \leq \min(r, s)$. Consider the subspace $I^t_{r,s}$ of the quotient space $J^t_{r,s}/J^{t+1}_{r,s}$ spanned by $Y^t_{r,s}$, where $Y^t_{r,s}$ denote the set of images of the diagrams with exactly t horizontal strands at the top connecting the $(r+1-k)$-th vertex to the $(r + k)$-th vertex for each $k = 1, 2 \ldots, t$. We have $\dim I^t_{r,s} = \binom{r}{t}\binom{s}{t} t!(r - t)!(s - t)!$. Then the space $I^t_{r,s}$ has a $(B_{r,s}(\delta), \mathbb{C}[\mathfrak{S}_{r-t}] \otimes \mathbb{C}[\mathfrak{S}_{s-t}])$-bimodule structure given by the left multiplication and the right multiplication, respectively. Now we define

$$C_{r,s}(\lambda) := I^t_{r,s} \otimes_{\mathbb{C}[\mathfrak{S}_{r-t}]\otimes\mathbb{C}[\mathfrak{S}_{s-t}]} S(\lambda^L) \boxtimes S(\lambda^R),$$

where $S(\lambda^L)$ denotes the simple $\mathfrak{S}_{r-t}$-module parametrized the partition λ^L of $r - t$, and $S(\lambda^R)$ denotes the simple $\mathfrak{S}_{s-t}$-module parametrized the partition λ^R of $s-t$ (see, for example, [11]). Note that the bimodule $I^t_{r,s}$ is free over $\mathbb{C}[\mathfrak{S}_{r-t}] \otimes \mathbb{C}[\mathfrak{S}_{s-t}]$ with basis $X^t_{r,s}$ consisting

of elements in $Y^t_{r,s}$ in which no two vertical strands cross. The cardinality of $X^t_{r,s}$ is $\binom{r}{t}\binom{s}{t}t!$. Thus as vector spaces,

$$C_{r,s}(\lambda) = \bigoplus_{\tau \in X^t_{r,s}} \tau \otimes (S(\lambda^L) \boxtimes S(\lambda^R)).$$

For $\lambda \in \dot{\Lambda}_{r,s}$, the cell module $C_{r,s}(\lambda)$ has an irreducible head $D_{r,s}(\lambda)$ and the family $\left\{D_{r,s}(\lambda) \mid \lambda \in \dot{\Lambda}_{r,s}\right\}$ is the complete set of mutually non-isomorphic simple modules over $B_{r,s}(\delta)$ (see [3, Theorem 2.7]). The elements in $\Lambda_{r,s}$ are called the *weights*.

The cell modules are indecomposable. Moreover, we have

Lemma 1.2. ([2, Lemma 2.2]) For any $\lambda \in \Lambda_{r,s}$ we have

$$\mathrm{End}_{B_{r,s}(\delta)}(C_{r,s}(\lambda)) \simeq \mathbb{C}.$$

2. Jucys-Murphy elements and supersymmetric polynomials

2.1. Jucys-Murphy elements.

Definition 2.1. For each $1 \le k \le r+s$, we define

$$L_k := \begin{cases} \displaystyle\sum_{j=1}^{k-1}(j,k) & \text{if } \ 1 \le k \le r, \\ \displaystyle -\sum_{j=1}^{r} e_{j,k} + \sum_{j=r+1}^{k-1}(j,k) + \delta & \text{if } \ r+1 \le k \le r+s. \end{cases}$$

We call these L_k's the *Jucys-Murphy elements of* $B_{r,s}(\delta)$. In particular, $L_1 = 0$.

Remark 2.2. The above Jucys-Murphy elements are similar to those in [2] and those in [32], but different. Precisely speaking, the parameter δ does not appear in the definitions of Jucys-Murphy elements in [2, 32]. Actually, if we modify the definition of $\xi_1 \mathbf{1_a}$, which was defined as $0\mathbf{1_a}$ in [32, (2.7)], into

$$\xi_1 \mathbf{1_a} = \begin{cases} 0\mathbf{1_a} & \text{if } \ a_1 = \uparrow, \\ \delta\mathbf{1_a} & \text{if } \ a_1 = \downarrow, \end{cases}$$

then the elements $\xi_k \mathbf{1}_{\uparrow^r\downarrow^s}$ in [32, (2.7)] yields our element L_k of $B_{r,s}(\delta)$. This modification is due to Antonio Sartori.

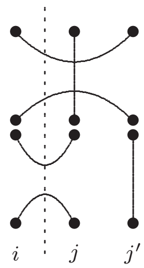
Note that there are so called *Jucys-Murphy-like elements* of walled Brauer algebras, introduced in [29], which are still different from ours.

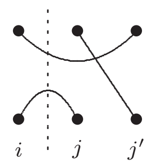
The following relations will be used frequently.

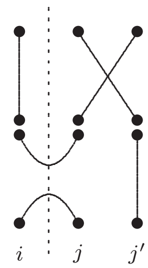
Lemma 2.3. *For all mutually distinct and admissible i, j, i', j', we have*
(1) $(i',j')e_{i,j} = e_{i,j}(i',j')$,
(2) $(i,i')e_{i,j} = e_{i',j}(i,i')$ *and* $(j,j')e_{i,j} = e_{i,j'}(j,j')$,
(3) $e_{i,j}e_{i',j'} = e_{i',j'}e_{i,j}$,
(4) $e_{i,j}e_{i,j'} = e_{i,j}(j,j')$ *and* $e_{i,j}e_{i'j} = e_{i,j}(i,i')$,

(5) $e_{i,j}e_{i',j'}(i,i') = e_{i,j}e_{i',j'}(j,j')$,

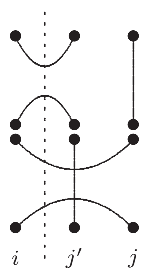
Proof. They can be checked by direct calculations on (r,s)-walled Brauer diagrams. For example, we can check the first equality in (4) as follows:

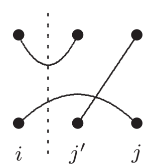
$e_{i,j}e_{i,j'} =$ 

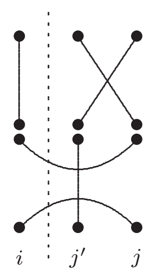
$=$ 

$=$ 

$= e_{i,j}(j,j')$ if $j < j'$,

$e_{i,j}e_{i,j'} =$ 

$=$ 

$=$ 

$= e_{i,j}(j,j')$ if $j' < j$.

Since the other relations can be checked similarly, we omit their proofs. □

The following proposition will play an important role in the rest of this paper. Note that similar relations in Brauer algebras appeared in [26, Proposition 2.3].

Proposition 2.4. *We have the following relations:*

(1) $(i,i+1)L_{i+1} - L_i(i,i+1) = 1$ *for* $i \neq r$,
(2) $L_{i+1}(i,i+1) - (i,i+1)L_i = 1$ *for* $i \neq r$,
(3) $e_{r,r+1}(L_r + L_{r+1}) = (L_r + L_{r+1})e_{r,r+1} = 0$,
(4) $(i,i+1)L_k = L_k(i,i+1)$ *for* $k \neq i, i+1$,
(5) $e_{r,r+1}L_k = L_k e_{r,r+1}$ *for* $k \neq r, r+1$.

Proof. (1) When $1 \leq i \leq r-1$, the relation (1) is well-known. Assume that $r+1 \leq i \leq r+s-1$. Then we have

$$\begin{aligned}
(i,i+1)L_{i+1} \quad &= (i,i+1)\Big(-\sum_{j=1}^{r} e_{j,i+1} + \sum_{j=r+1}^{i} (j,i+1) + \delta\Big) \\
&= -\sum_{j=1}^{r} e_{j,i}(i,i+1) + \sum_{j=r+1}^{i-1} (i,i+1)(j,i+1) + 1 + \delta(i,i+1) \\
&= \Big(-\sum_{j=1}^{r} e_{j,i} + \sum_{j=r+1}^{i-1} (j,i) + \delta\Big)(i,i+1) + 1 = L_i(i,i+1) + 1.
\end{aligned}$$

(2) It follows from (1) that

$$L_{i+1}(i,i+1) = (i,i+1)(L_i(i,i+1)+1)(i,i+1) = (i,i+1)L_i + 1.$$

(3) By direct calculations and Lemma 2.3 (4), we get

$$\begin{aligned} e_{r,r+1}(L_r + L_{r+1}) &= e_{r,r+1}\Big(\sum_{j=1}^{r-1}(j,r) - \sum_{j=1}^{r} e_{j,r+1} + \delta\Big) \\ &= \sum_{j=1}^{r-1} e_{r,r+1}(j,r) - \sum_{j=1}^{r-1} e_{r,r+1}(j,r) - \delta e_{r,r+1} + \delta e_{r,r+1} = 0. \end{aligned}$$

Similarly, by Lemma 2.3 (2) and (4), we have

$$\begin{aligned} (L_r + L_{r+1})e_{r,r+1} &= \Big(\sum_{j=1}^{r-1}(j,r) - \sum_{j=1}^{r} e_{j,r+1} + \delta\Big)e_{r,r+1} \\ &= \sum_{j=1}^{r-1} e_{j,r+1}(j,r) - \sum_{j=1}^{r-1} e_{j,r+1}(j,r) - \delta e_{r,r+1} + \delta e_{r,r+1} = 0. \end{aligned}$$

(4) For $i \geq k+1$, it is trivial that $L_k(i,i+1) = (i,i+1)L_k$. For $i < k-1 < k \leq r$, it is well-known that $(i,i+1)$ commutes with L_k. Assume that $i < r < k$. Then from Lemma 2.3 (1),(2) and direct calculations, we have

$$\begin{aligned} (i,i+1)L_k &= (i,i+1)\Big(-\sum_{j=1}^{r} e_{j,k} + \sum_{j=r+1}^{k-1}(j,k) + \delta\Big) \\ &= -\sum_{\substack{j=1 \\ j\neq i,i+1}}^{r} e_{j,k}(i,i+1) - e_{i+1,k}(i,i+1) - e_{i,k}(i,i+1) \\ &\quad + \sum_{j=r+1}^{k-1}(j,k)(i,i+1) + \delta(i,i+1) \\ &= \Big(-\sum_{j=1}^{r} e_{j,k} + \sum_{j=r+1}^{k-1}(j,k) + \delta\Big)(i,i+1) = L_k(i,i+1). \end{aligned}$$

Assume that $r < i < k-1$. Then we can check that

$$\begin{aligned} (i,i+1)L_k &= (i,i+1)\Big(-\sum_{j=1}^{r} e_{j,k} + \sum_{j=r+1}^{k-1}(j,k) + \delta\Big) \\ &= -\sum_{j=1}^{r} e_{j,k}(i,i+1) + \sum_{\substack{j=r+1 \\ j\neq i,i+1}}^{k-1}(j,k)(i,i+1) + \delta(i,i+1) \\ &\quad +(i,i+1)(i,k) + (i,i+1)(i+1,k) \end{aligned}$$

$$
\begin{aligned}
&= -\sum_{j=1}^{r} e_{j,k}(i,i+1) + \sum_{\substack{j=r+1\\ j\neq i,i+1}}^{k-1} (j,k)(i,i+1) + \delta(i,i+1)\\
&\quad +(i+1,k)(i,i+1) + (i,k)(i,i+1)\\
&= \Big(-\sum_{j=1}^{r} e_{j,k} + \sum_{j=r+1}^{k-1} (j,k) + \delta\Big)(i,i+1) = L_k(i,i+1).
\end{aligned}
$$

(5) If $k < r$, it is trivial that $e_{r,r+1}L_k = L_k e_{r,r+1}$. Let $k \geq r+2$. Then we have

$$
\begin{aligned}
e_{r,r+1}L_k \qquad &= e_{r,r+1}\Big(-\sum_{j=1}^{r} e_{j,k} + \sum_{j=r+1}^{k-1} (j,k) + \delta\Big)\\
&= -\sum_{j=1}^{r-1} e_{r,r+1}e_{j,k} - e_{r,r+1}(r+1,k) + \sum_{j=r+1}^{k-1} e_{r,r+1}(j,k) + \delta e_{r,r+1}\\
&= -\sum_{j=1}^{r-1} e_{j,k}e_{r,r+1} + \sum_{j=r+2}^{k-1} e_{r,r+1}(j,k) + \delta e_{r,r+1}\\
&= \Big(-\sum_{j=1}^{r} e_{j,k} + \sum_{j=r+1}^{k-1} (j,k) + \delta\Big)e_{r,r+1} = L_k e_{r,r+1},
\end{aligned}
$$

as desired. In the fourth equality, we use $e_{r,k}e_{r,r+1} = e_{r,k}(r+1,k) = (r+1,k)e_{r,r+1}$ obtained by Lemma 2.3 (2) and (4). $\square$

Remark 2.5. For each $a \in \mathbb{C}$, we have variants of the Jucys-Murphy elements: set

$$
L_k^a := \begin{cases} L_k + a & \text{if } \ 1 \leq k \leq r,\\ L_k - a & \text{if } \ r+1 \leq k \leq r+s. \end{cases}
$$

It is easy to see that the above proposition holds with L_k^a instead of L_k. All the other parts of the rest of this paper will be also valid, after a small modification.

Proposition 2.6. *The elements L_k's are commuting to each other.*

Proof. It is well-known that the elements $L_1, \ldots, L_r$ are commuting to each other. Let B_r be the subalgebra of $B_{r,s}(\delta)$ generated by $s_1, \ldots, s_{r-1}$ and let B_{r+a} be the subalgebra generated by $s_1, \ldots, s_{r-1}, e_{r,r+1}, s_{r+1}, \ldots s_{r+a-1}$ for $1 \leq a \leq s$. Then we have $L_1, \ldots, L_{r+a-1}, L_{r+a} \in B_{r+a}$ for $0 \leq a \leq s$. On the other hand, by Proposition 2.4 (4) and (5), we know that the element L_{r+a} commutes with the generators of B_{r+a-1} for each $1 \leq a \leq s$. Therefore, L_{r+a} commutes with $L_1, \ldots, L_{r+a-1}$, as desired. $\square$

Proposition 2.7. *For each $k \in \mathbb{Z}_{\geq 0}$, the element*

$$
L_1^k + \cdots + L_r^k + (-1)^{k+1}(L_{r+1}^k + \cdots + L_{r+s}^k)
$$

belongs to the center of $B_{r,s}(\delta)$.

Proof. From Proposition 2.4 (1) and (2), we have

$$(i,i+1)(L_i+L_{i+1}) = (L_i+L_{i+1})(i,i+1),$$
$$(i,i+1)(L_iL_{i+1}) = (L_{i+1}L_i)(i,i+1) = (L_iL_{i+1})(i,i+1)$$

for $i \neq r$. Thus $(i,i+1)$ commutes with every symmetric polynomials in L_i and L_{i+1}. In particular, it commutes with the power sum symmetric polynomials $L_i^k + L_{i+1}^k$ $(k \geq 0)$. Combining this fact with Proposition 2.4 (4), it follows that

$$(i,i+1)(L_1^k+\cdots+L_r^k) = (L_1^k+\cdots+L_r^k)(i,i+1)$$
$$(i,i+1)(L_{r+1}^k+\cdots+L_{r+s}^k) = (L_{r+1}^k+\cdots+L_{r+s}^k)(i,i+1)$$

for all $i \neq r$.

From Proposition 2.4 (3), we obtain

$$e_{r,r+1}(L_r^k+(-1)^{k+1}L_{r+1}^k) = 0 = (L_r^k+(-1)^{k+1}L_{r+1}^k)e_{r,r+1}.$$

Hence, by using Proposition 2.4 (5), we conclude that $L_1^k+\cdots+L_r^k+(-1)^{k+1}(L_{r+1}^k+\cdots+L_{r+s}^k)$ is in the center of $B_{r,s}(\delta)$, since it commutes with all the generators. □

When $k = 1$, the above was shown in [2, Lemma 2.1]. Indeed, the element $z_{r,s}$ in [2, Lemma 2.1] is the same as $L_1+\cdots+L_{r+s}-s\delta$.

2.2. **Supersymmetric polynomials.** In this section we recall the notion of supersymmetric polynomials. For details on supersymmetric polynomials, see for example, [23].

Definition 2.8. Let r, s be nonnegative integers. We say that an element p in the polynomial ring $\mathbb{C}[x_1,\ldots,x_r,y_1\ldots,y_s]$ is *supersymmetric* if

(1) p is *doubly symmetric*; i.e, it is symmetric in $x_1,\ldots,x_r$ and $y_1,\ldots,y_s$ separately,
(2) p satisfies the *cancellation property*; i.e., the substitution $x_r = -y_1 = t$ yields a polynomial in $x_1,\ldots,x_{r-1},y_2,\ldots,y_s$ which is independent of t.

We denote $S_{r,s}[x;y]$ the set of supersymmetric polynomials in $x_1,\ldots,x_r,y_1\ldots,y_s$. It is a $\mathbb{C}$-subalgebra of $\mathbb{C}[x_1,\ldots,x_r,y_1\ldots,y_s]^{\mathfrak{S}_r\times\mathfrak{S}_s}$, the algebra of doubly symmetric polynomials.

For $k \geq 0$, the *k-th power sum supersymmetric polynomial* is given by

$$p_k(x_1,\ldots,x_r,y_1,\ldots,y_s) := x_1^k+\cdots+x_r^k+(-1)^{k+1}(y_1^k+\cdots+y_s^k).$$

It is easy to see that p_k belongs to $S_{r,s}[x;y]$. In [36], Stembridge showed that $S_{r,s}[x;y]$ is generated by $\{p_k \mid k \geq 0\}$. Hence the following is an immediate consequence of Proposition 2.7.

Corollary 2.9. *For every supersymmetric polynomial p in $S_{r,s}[x;y]$, the element*

$$p(L_1,\ldots,L_{r+s})$$

belongs to the center $Z(B_{r,s}(\delta))$ of $B_{r,s}(\delta)$.

Remark 2.10. If we take the modification in Remark 2.2 and focus on the case of $B_{r,s}(\delta)$, then the above corollary corresponds to [32, Corollary 7.2], Note that in [32, Corollary 7.2] they used a description of the center of the degenerate affine walled Brauer algebra.

The *elementary supersymmetric polynomials*

$$e_k(x_1, \ldots, x_r, y_1, \ldots, y_s) \quad (k \in \mathbb{Z}_{\geq 0})$$

are given by the generating function

$$\sum_{k=0}^{\infty} e_k(x_1, \ldots, x_r, y_1, \ldots, y_s) z^k = \frac{\prod_{i=1}^{r}(1 + x_i z)}{\prod_{j=1}^{s}(1 - y_j z)}.$$

It is also known that $\{e_k \mid k \in \mathbb{Z}_{\geq 0}\}$ generates the ring of supersymmetric polynomials ([36, Corollary]). Then the lemma below follows immediately.

Lemma 2.11. *Let $(a_1, \ldots, a_r, b_1, \ldots, b_s)$ and $(c_1, \ldots, c_r, d_1, \ldots, d_s)$ be elements in $\mathbb{C}^{r+s}$. Then the followings are equivalent.*

(1) *For every supersymmetric polynomial $p \in S_{r,s}[x; y]$, we have*

$$p(a_1, \ldots, a_r, b_1, \ldots, b_s) = p(c_1, \ldots, c_r, d_1, \ldots, d_s).$$

(2) *We have an equality*

$$\frac{\prod_{i=1}^{r}(1 + a_i z)}{\prod_{j=1}^{s}(1 - b_j z)} = \frac{\prod_{i=1}^{r}(1 + c_i z)}{\prod_{j=1}^{s}(1 - d_j z)}$$

of rational functions in z.

For a partition $\mu = (\mu_1, \mu_2, \ldots)$, a filling of μ with entries $1, \ldots, |\mu|$ is called a *standard tableau of shape* μ, when the entries in each row and each column are strictly increasing, from left to right and from top to bottom, respectively. Let $\mathfrak{t}^\mu$ be the standard tableau such that the entries $1, 2, \ldots, |\mu|$ appear in increasing order from left to right along successive rows. Recall that, for a box u in $[\mu]$, *the content of* u is given by the integer $b - a$, where u is located (a, b)-position in $[\mu]$. For $1 \leq i \leq |\mu|$, we define $\mathrm{cont}(\mu, i)$ to be the content of the box in μ with entry i in the tableau $\mathfrak{t}^\mu$. It is called the *content of* μ *at* i. Note that the multiset $\{\mathrm{cont}(\mu, i) \mid 1 \leq i \leq |\mu|\}$ determines the Young diagram $[\mu]$ and hence the partition μ.

For each $\lambda \in \Lambda_{r,s}^t$, set

$$c(\lambda, i) := \begin{cases} \mathrm{cont}(\lambda^L, i) & \text{if } 1 \leq i \leq r - t, \\ 0 & \text{if } r - t + 1 \leq i \leq r + t, \\ \mathrm{cont}(\lambda^R, i - r - t) + \delta & \text{if } r + t + 1 \leq i \leq r + s. \end{cases}$$

Lemma 2.12. *Assume that $B_{r,s}(\delta)$ is semisimple. If $\lambda \neq \mu$ for $\lambda, \mu \in \Lambda_{r,s}$, then there is a supersymmetric polynomial $p^{\lambda,\mu}$ such that*

$$p^{\lambda,\mu}((c(\lambda, i))_{1 \leq i \leq r+s}) \neq p^{\lambda,\mu}((c(\mu, i))_{1 \leq i \leq r+s}).$$

Remark 2.13. If we combine [3, Corollary 7.7] with Lemma 5.2 in the last section, then the above lemma follows. But we include the following proof for completeness.

Proof. Let $\lambda \in \Lambda_{r,s}^t$, $\mu \in \Lambda_{r,s}^{t'}$ for some $0 \leq t, t' \leq \min(r, s)$.

Suppose that

$$p((c(\lambda, i))_{1 \leq i \leq r+s}) = p((c(\mu, i))_{1 \leq i \leq r+s})$$

for every $p \in S_{r,s}[x;y]$. Equivalently,

$$\frac{\prod_{i=1}^{r-t}(1+\mathrm{cont}(\lambda^L,i)z)}{\prod_{j=r+t+1}^{r+s}(1-(\mathrm{cont}(\lambda^R,j)+\delta)z)} = \frac{\prod_{i=1}^{r-t'}(1+\mathrm{cont}(\mu^L,i)z)}{\prod_{j=r+t'+1}^{r+s}(1-(\mathrm{cont}(\mu^R,j)+\delta)z)}. \tag{2.1}$$

We shall show that $\lambda = \mu$ for all cases in Proposition 1.1.

Case (1) : Assume that $r = 0$ or $s = 0$. Then we have $t = t' = 0$.

Assume that $s = 0$. Then $\lambda^R = \mu^R = \emptyset$ and we have

$$\prod_{i=1}^{r}(1+\mathrm{cont}(\lambda^L,i)z) = \prod_{i=1}^{r}(1+\mathrm{cont}(\mu^L,i)z).$$

It follows that the multisets of contents of λ^L and μ^L are the same. Thus $\lambda^L = \mu^L$. The proof for the case $r = 0$ is the same.

Case (2), (3) : Assume that $\delta \notin \mathbb{Z}$ or $|\delta| > r+s-2$. We may further assume that $r > 0$ and $s > 0$ so that $r+s-2 \geq 0$. Now we have

$$\begin{gathered}\prod_{i=1}^{r-t}(1+\mathrm{cont}(\lambda^L,i)z) = \prod_{i=1}^{r-t'}(1+\mathrm{cont}(\mu^L,i)z) \quad \text{and} \\ \prod_{j=r+t+1}^{r+s}(1-(\mathrm{cont}(\lambda^R,j)+\delta)z) = \prod_{j=r+t'+1}^{r+s}(1-(\mathrm{cont}(\mu^R,j)+\delta)z).\end{gathered} \tag{2.2}$$

Indeed if $\delta \notin \mathbb{Z}$ then it is trivial. Otherwise, we have

$$\begin{aligned}|\mathrm{cont}(\lambda^L,i)+\mathrm{cont}(\lambda^R,j)| &\leq (r-t-1)+(s-t-1) \leq r+s-2 < |\delta|,\\ |\mathrm{cont}(\mu^L,i)+\mathrm{cont}(\mu^R,j)| &\leq (r-t'-1)+(s-t'-1) \leq r+s-2 < |\delta|.\end{aligned}$$

It follows that the multiset of *nonzero* contents of λ^L is the same as the one of μ^L and the multiset of contents of λ^R *which are not equal to* $-\delta$ is the same as the one of μ^R.

Note that

$$\left\{j \mid \mathrm{cont}(\lambda^R,i) = -\delta\right\} = \left\{j \mid \mathrm{cont}(\mu^R,j) = -\delta\right\} = \emptyset.$$

Indeed it it trivial if $\delta \notin \mathbb{Z}$ and otherwise we have

$$\begin{aligned}|\mathrm{cont}(\lambda^R,i)| &\leq s-t-1 \leq r+s-t-2 \leq r+s-2 < |\delta|,\\ |\mathrm{cont}(\mu^R,i)| &\leq s-t'-1 \leq r+s-t'-2 \leq r+s-2 < |\delta|.\end{aligned}$$

Hence, observing the degree of (2.2), we have $t = t'$. Thus the multiset of contents of λ^L is the same as the one of μ^L and the multiset of contents of λ^R is the same as the one of μ^R. It follows that $\lambda = \mu$.

Case (4) : Assume $\delta = 0$ and $(r,s) \in \{(1,2),(2,1),(1,3),(3,1)\}$. If $(r,s) = (3,1)$, then we have

$$\left\{(c(\lambda,i))_{1\leq i\leq 4} \mid \lambda \in \dot{\Lambda}_{3,1}\right\} = \left\{(0,1,2,0),(0,1,-1,0),(0,-1,-2,0),(0,1,0,0),(0,-1,0,0)\right\}.$$

It is easy to observe that (2.1) holds if and only if $\lambda = \mu$.

The cases $(r,s) \in \{(1,2),(2,1),(1,3)\}$ can be checked in a similar way. $\square$

3. Center of the walled Brauer algebra

By Proposition 2.7, the power sum supersymmetric polynomials in the Jucys-Murphy elements belong to the center of $B_{r,s}(\delta)$. Hence they act by scalar multiplications on a cell module by Lemma 1.2. More precisely, we have

Proposition 3.1. *For $\lambda \in \Lambda^t_{r,s}$ and for $k \geq 0$, we have*

$$p_k(L_1, \ldots, L_r, L_{r+1}, \ldots, L_{r+s}) = \sum_{i=1}^{r-t} \mathrm{cont}(\lambda^L, i)^k + (-1)^{k+1} \sum_{i=1}^{s-t} (\mathrm{cont}(\lambda^R, i) + \delta)^k$$

on the cell module $C_{r,s}(\lambda)$.

When $k = 1$, the above can be obtained from [2, Lemma 2.3] directly, because $z_{r,s}$ in [2, Lemma 2.3] is the same as $L_1 + \cdots + L_{r+s} - s\delta$.

To prove Proposition 3.1, we need the following lemma. We use the same technique in [2, Lemma 2.3]. For reader's convenience, we include a proof.

Lemma 3.2. *For each $1 \leq t \leq \min(r, s)$, set*

$$\tau_t := e_{r,r+1} e_{r-1,r+2} \cdots e_{r-t+1,r+t} \in B_{r,s}(\delta). \tag{3.1}$$

We have

(1) $(L_r + L_{r+1})\tau_t = (L_{r-1} + L_{r+2})\tau_t = \cdots = (L_{r-t+1} + L_{r+t})\tau_t = 0$,
(2) $\Big(- \sum_{i=r-t+1}^{r} e_{i,j} + \sum_{i=r+1}^{r+t} (i,j) \Big) \tau_t = 0$ *for any $j \geq r + t + 1$.*

Proof. (1) Let $I := \{1, \ldots, r-t\}$, $J = \{r+t+1, \ldots, r+s\}$ and $K := \{r-t+1, \ldots, r\}$. For $a \in K$, let $\widetilde{a} := 2r + 1 - a$. We shall show that

$$(L_a + L_{\widetilde{a}})\tau_t = 0 \qquad \text{for all } a \in K.$$

We can obtain the following relations using Lemma 2.3:

(1) $((i,a) - e_{i,\widetilde{a}})e_{a,\widetilde{a}} = 0$ for $i \in I$,
(2) $((i,a) - e_{i,\widetilde{a}})e_{i,\widetilde{i}}e_{a,\widetilde{a}} = 0$ for $i \in K, i < a$,
(3) $(-e_{a,\widetilde{a}} + \delta)e_{a,\widetilde{a}} = 0$,
(4) $(-e_{i,\widetilde{a}} + (\widetilde{i}, \widetilde{a}))e_{i,\widetilde{i}}e_{a,\widetilde{a}} = 0$ for $i \in K, i > a$.

More precisely, we can check that

(1) $((i,a) - e_{i,\widetilde{a}})e_{a,\widetilde{a}} = e_{i,\widetilde{a}}(i,a) - e_{i,\widetilde{a}}(i,a) = 0$,
(2) $((i,a) - e_{i,\widetilde{a}})e_{i,\widetilde{i}}e_{a,\widetilde{a}} = e_{a,\widetilde{i}}(i,a)e_{a,\widetilde{a}} - e_{i,\widetilde{a}}(\widetilde{i},\widetilde{a})e_{a,\widetilde{a}} = e_{a,\widetilde{i}}e_{i,\widetilde{a}}(i,a) - e_{i,\widetilde{a}}e_{a,\widetilde{i}}(\widetilde{i},\widetilde{a}) = 0$,
(3) $(-e_{a,\widetilde{a}} + \delta)e_{a,\widetilde{a}} = -\delta e_{a,\widetilde{a}} + \delta e_{a,\widetilde{a}} = 0$,
(4) $(-e_{i,\widetilde{a}} + (\widetilde{i},\widetilde{a}))e_{i,\widetilde{i}}e_{a,\widetilde{a}} = -e_{i,\widetilde{a}}(\widetilde{a},\widetilde{i})e_{a,\widetilde{a}} + e_{i,\widetilde{a}}(\widetilde{i},\widetilde{a})e_{a,\widetilde{a}} = -e_{i,\widetilde{a}}e_{a,\widetilde{i}}(\widetilde{a},\widetilde{i}) + e_{i,\widetilde{a}}e_{a,\widetilde{i}}(\widetilde{i},\widetilde{a}) = 0$.

We separate $L_a = \sum_{i \in I}(i,a) + \sum_{i<a, i \in K}(i,a)$, and

$$L_{\widetilde{a}} = -\sum_{i \in I} e_{i,\widetilde{a}} - \sum_{i<a, i \in K} e_{i,\widetilde{a}} - e_{a,\widetilde{a}} - \sum_{i>a, i \in K} e_{i,\widetilde{a}} + \sum_{i>a, i \in K} (\widetilde{i}, \widetilde{a}) + \delta.$$

Note that all the factors in (3.1) are commuting to each other by Lemma 2.3 (3). Using the relations (1)-(4), we have $(L_a + L_{\widetilde{a}})\tau_t = 0$.

(2) For $i \in K$ and $j \in J$, we have

$$(-e_{i,j} + (\widetilde{i}, j))e_{i,\widetilde{i}} = -e_{i,j}(j, \widetilde{i}) + e_{i,j}(\widetilde{i}, j) = 0.$$

Thus we have

$$(-e_{i,j} + (\widetilde{i}, j))\tau_t = 0.$$

It follows that

$$\Big(- \sum_{i=r-t+1}^{r} e_{i,j} + \sum_{i=r+1}^{r+t} (i, j)\Big)\tau_t = \sum_{i=r-t+1}^{r} (-e_{i,j} + (\widetilde{i}, j))\tau_t = 0.$$

□

Now we prove Proposition 3.1.

Proof. In $C_{r,s}(\lambda)$, the element $\tau \otimes v$ generates $C_{r,s}(\lambda)$ as a $B_{r,s}(\delta)$-module, where τ is the image of $\tau_t = e_{r,r+1}e_{r-1,r+2} \cdots e_{r-t+1,r+t}$ $(t \geq 1)$ or $\mathbf{1}$ $(t = 0)$ in $J^t_{r,s}/J^{t+1}_{r,s}$ and v is a non-zero vector in $S(\lambda^L) \boxtimes S(\lambda^R)$. Since $p_k(L_1, \dots, L_{r+s})$ is central, it is enough to show that

$$\begin{aligned} &p_k(L_1, \dots, L_r, L_{r+1}, \dots, L_{r+s})(\tau \otimes v) \\ &\qquad = \Big(\sum_{i=1}^{r-t} \mathrm{cont}(\lambda^L, i)^k + (-1)^{k+1} \sum_{i=1}^{s-t} (\mathrm{cont}(\lambda^R, i) + \delta)^k \Big)(\tau \otimes v), \end{aligned}$$

for all $k \geq 1$. By Lemma 3.2 (1), we know that

$$(L^k_{r-t+1} + \cdots + L^k_r + (-1)^{k+1}(L^k_{r+1} + \cdots + L^k_{r+t}))\tau_t = 0.$$

Hence it is enough to show that

$$\Big(\sum_{j=1}^{r-t} L^k_j \Big)\big(\tau \otimes v\big) = \Big(\sum_{i=1}^{r-t} \mathrm{cont}(\lambda^L, i)^k \Big)\tau \otimes v \quad \text{and,}$$

$$\Big(\sum_{j=r+t+1}^{r+s} L^k_j \Big)\big(\tau \otimes v\big) = \Big(\sum_{j=1}^{s-t} (\mathrm{cont}(\lambda^R, j) + \delta)^k \Big)\tau \otimes v.$$

Note that $(L^k_1 + \cdots + L^k_{r-t})\tau = \tau(L^k_1 + \cdots + L^k_{r-t})$ and $L^k_1 + \cdots + L^k_{r-t} \in \mathbb{C}[\mathfrak{S}_{r-t}] \otimes \mathbf{1} \subset \mathbb{C}[\mathfrak{S}_{r-t}] \otimes \mathbb{C}[\mathfrak{S}_{s-t}]$. Moreover, $L^k_1 + \cdots + L^k_{r-t}$ corresponds to a symmetric polynomial in the Jucys-Murphy elements of $\mathbb{C}[\mathfrak{S}_{r-t}]$ under the isomorphism $\langle s_1, \dots, s_{r-t-1} \rangle \simeq \mathbb{C}[\mathfrak{S}_{r-t}]$. Thus we obtain

$$\begin{aligned} &(L^k_1 + \cdots + L^k_{r-t})\tau \otimes v = \tau(L^k_1 + \cdots + L^k_{r-t}) \otimes v \\ &= \tau \otimes (L^k_1 + \cdots + L^k_{r-t})v = \tau \otimes \big(\sum_{i=1}^{r-t} \mathrm{cont}(\lambda^L, i)^k \big)v, \end{aligned}$$

where the last equality follows from the fact that a symmetric polynomial in Jucys-Murphy elements of a symmetric group acts on a simple module $S(\mu)$ associated with a partition μ by

the scalar multiplication, which is given by the evaluation of the polynomial at the contents of μ ([24]). See also [5, Theorem 1.1].

When $1 \leq i \leq r-t$, and $r+t+1 \leq j \leq r+s$, we get $e_{i,j}\tau = 0$ on $J^t_{r,s}/J^{t+1}_{r,s}$ because $e_{i,j}$ makes another horizontal strand. Combining this and Lemma 3.2 (2), we have

$$L_j\tau = \Big(\sum_{i=r+t+1}^{j-1}(i,j)+\delta\Big)\tau = \tau\Big(\sum_{i=r+t+1}^{j-1}(i,j)+\delta\Big) \qquad \text{for } r+t+1 \leq j \leq r+s$$

on $J^t_{r,s}/J^{t+1}_{r,s}$. Note that we used the fact that τ commutes with (p,q), if $p,q \geq r+t+1$. By repeating this procedure, we obtain

$$L_j^k\tau = \tau\Big(\sum_{i=r+t+1}^{j-1}(i,j)+\delta\Big)^k \qquad \text{for } r+t+1 \leq j \leq r+s$$

on $J^t_{r,s}/J^{t+1}_{r,s}$. Now we have

$$\Big(\sum_{j=r+t+1}^{r+s} L_j^k\Big)(\tau\otimes v) = \sum_{j=r+t+1}^{r+s} \tau\Big(\sum_{i=r+t+1}^{j-1}(i,j)+\delta\Big)^k \otimes v = \tau\otimes\Big(\sum_{j=r+t+1}^{r+s}\Big(\sum_{i=r+t+1}^{j-1}(i,j)+\delta\Big)^k\Big)v,$$

because $\sum_{i=r+t+1}^{j-1}(i,j)+\delta \in \mathbf{1}\otimes\mathbb{C}[\mathfrak{S}_{s-t}] \subset \mathbb{C}[\mathfrak{S}_{r-t}]\otimes\mathbb{C}[\mathfrak{S}_{s-t}]$. Since the element

$$\sum_{j=r+t+1}^{r+s}\Big(\sum_{i=r+t+1}^{j-1}(i,j)+\delta\Big)^k$$

is a symmetric polynomial in the Jucys-Murphy elements of the subalgebra $\mathbf{1}\otimes\mathbb{C}[\mathfrak{S}_{s-t}]$ under the isomorphism $\langle s_{r+t+1},\dots,s_{r+s-1}\rangle \simeq \mathbb{C}[\mathfrak{S}_{s-t}]$, it follows again by [24] (see also [5, Theorem 1.1]) that

$$\Big(\sum_{j=r+t+1}^{r+s}\Big(\sum_{i=r+t+1}^{j-1}(i,j)+\delta\Big)^k\Big)v = \Big(\sum_{j=1}^{s-t}(\mathrm{cont}(\lambda^R,j)+\delta)^k\Big)v.$$

Therefore, we obtain the desired assertion. $\square$

Corollary 3.3. *Let f be a supersymmetric polynomial in $S_{r,s}[x;y]$. Then we have*

$$f(L_1,\dots,L_r,L_{r+1},\dots,L_{r+s}) = f(c(\lambda,1),\dots,c(\lambda,r),c(\lambda,r+1),\dots,c(\lambda,r+s))$$

on $C_{r,s}(\lambda)$ for every $\lambda\in\Lambda_{r,s}$.

Proof. It follows immediately from the above proposition and the fact that $S_{r,s}[x;y]$ is generated by the power sum supersymmetric polynomials ([36]). $\square$

The proof of the following lemma is identical to the argument in [20, Theorem 3.3]. We reproduce it here in a slightly more general setting for reader's convenience.

Lemma 3.4. ([20, Theorem 3.3]) Let S be a $\mathbb{C}$-subalgebra of the polynomial ring $\mathbb{C}[X_1,\dots,X_m]$ and let

$$(k_{11},\dots,k_{1m}),\dots,(k_{n1},\dots,k_{nm}) \tag{3.2}$$

be n sequences of elements in $\mathbb{C}$ for some positive integer n. Assume that

$$\text{for each } 1 \le i \ne j \le n, \text{ there exists an element } p \text{ in } S \text{ such that } p(i) \ne p(j), \tag{3.3}$$

where $p(i)$ denotes the value $p(k_{i1}, \dots, k_{im})$.

Then there exists a family of elements $p_1, \dots, p_n$ in S such that

$$\begin{vmatrix} p_1(1) & p_1(2) & \cdots & p_1(n) \\ p_2(1) & p_2(2) & \cdots & p_2(n) \\ \cdots & \cdots & \cdots & \cdots \\ p_n(1) & p_n(2) & \cdots & p_n(n) \end{vmatrix} \ne 0.$$

Proof. We will proceed by induction on n. Let $(k_1, k_2, \dots, k_m)$ be a sequence of elements in $\mathbb{C}$. There exists an element p such that $p(k_1, k_2, \dots, k_m) \ne 0$. For example, we can take a nonzero constant polynomial as p.

Now assume that $n > 1$ and that the assertion holds for all $1 \le i \le n-1$. Consider the first $n-1$ sequences

$$(k_{11}, \dots, k_{1m}), \dots, (k_{n-11}, \dots, k_{n-1m})$$

of (3.2). Then, for $1 \le i \ne j \le n-1$ there exists a polynomial $p \in S$ such that $p(i) \ne p(j)$, by the assumption (3.3). Now, by the induction hypothesis, we assume that there exists a family of polynomials $p_1, \dots, p_{n-1}$ in S such that

$$\begin{vmatrix} p_1(1) & p_1(2) & \cdots & p_1(n-1) \\ p_2(1) & p_2(2) & \cdots & p_2(n-1) \\ \cdots & \cdots & \cdots & \cdots \\ p_{n-1}(1) & p_{n-1}(2) & \cdots & p_{n-1}(n-1) \end{vmatrix} \ne 0. \tag{3.4}$$

Applying elementary row operations to the above matrix, we may further assume that

$$p_i(j) = 0 \ (1 \le j < i \le n-1), \quad \text{and} \quad p_i(i) = 1 \ (1 \le i \le n-1) \tag{3.5}$$

(see, [20, Lemma 3.5]). In particular, the determinant of the above matrix is 1.

Suppose that for every element $p \in S$ we have

$$d(p) := \begin{vmatrix} p_1(1) & p_1(2) & \cdots & p_1(n-1) & p_1(n) \\ p_2(1) & p_2(2) & \cdots & p_2(n-1) & p_2(n) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{n-1}(1) & p_{n-1}(2) & \cdots & p_{n-1}(n-1) & p_{n-1}(n) \\ p(1) & p(2) & \cdots & p(n-1) & p(n) \end{vmatrix} = 0.$$

Then, by (3.4), (3.5) and the determinant expansion by minors, we have

$$p(n) = k_1 p(1) + k_2 p(2) + \cdots + k_{n-1} p(n-1),$$

for some $k_j \in \mathbb{C}$, which is independent of p for $1 \le j \le n-1$. Note that $p_{n-1}p$ is also in S so that we have

$$p_{n-1}p(n) = k_1 p_{n-1}p(1) + k_2 p_{n-1}p(2) + \cdots + k_{n-1} p_{n-1}p(n-1) = k_{n-1}p(n-1),$$

since $p_{n-1}(j) = 0$ for $1 \le j \le n-2$ and $p_{n-1}(n-1) = 1$.

Assume that $p_{n-1}(n) \neq 0$. Then we have

$$p(n) = \frac{k_{n-1}}{p_{n-1}(n)} p(n-1),$$

for all polynomial $p \in S$. Taking a nonzero constant polynomial $p \in S$, we have $\dfrac{k_{n-1}}{p_{n-1}(n)} = 1$ so that $p(n) = p(n-1)$ for all polynomials p in S. It is a contradiction to the assumption (3.3) for $i = n-1, j = n$.

Thus we have $p_{n-1}(n) = 0$. It implies that $k_{n-1}p(n-1) = 0$. Taking a nonzero constant polynomial p, we have $k_{n-1} = 0$. Repeating this process similarly with p_j instead of p_{n-1}, we have $k_j = 0$ for $j = 1, \dots, n-1$ and hence $p(n) = 0$. But it is impossible, since p is arbitrary. Thus we conclude that there exists an element p in S such that $d(p) \neq 0$. $\square$

Now we present our main theorem.

Theorem 3.5. *If the walled Brauer algebras $B_{r,s}(\delta)$ is semisimple, then the supersymmetric polynomials in $L_1, \dots, L_{r+s}$ generate the center of $B_{r,s}(\delta)$.*

Proof. By Lemma 2.12, we can apply the Proposition 3.4 to the case $S = S_{r,s}[x; y]$ with the sequences

$$\left\{ (c(\lambda, i))_{1 \le i \le r+s} \mid \lambda \in \dot{\Lambda}_{r,s} \right\}.$$

Thus we obtain a set of supersymmetric polynomials $\left\{ p_\lambda \in S_{r,s}[x; y] \mid \lambda \in \dot{\Lambda}_{r,s} \right\}$ such that the matrix

$$\big((p_\lambda(\mu)) \big)_{\lambda, \mu \in \dot{\Lambda}_{r,s}}$$

is nonsingular, where $p_\lambda(\mu) := p_\lambda(c(\mu, 1), \dots, c(\mu, r+s))$.

Assume that there is a family of complex numbers $\left\{ a_\lambda \in \mathbb{C} \mid \lambda \in \dot{\Lambda}_{r,s} \right\}$ such that

$$\sum_\lambda a_\lambda p_\lambda(L_1, \dots, L_{r+s}) = 0.$$

By Corollary 3.3, we have

$$\sum_\lambda a_\lambda p_\lambda(\mu) = 0 \quad \text{for all} \quad \mu \in \dot{\Lambda}_{r,s}.$$

Hence $a_\lambda = 0$ for all $\lambda \in \dot{\Lambda}_{r,s}$ so that

$$p_\lambda(L_1, \dots, L_{r+s}) \quad (\lambda \in \dot{\Lambda}_{r,s})$$

are linearly independent.

On the other hand, if $B_{r,s}(\delta)$ is semisimple, the dimension of the center is the same as the number of the isomorphism classes of simple modules, and hence it is identical to the cardinality of $\dot{\Lambda}_{r,s}$. Thus we conclude that $\left\{ p_\lambda(L_1, \dots, L_{r+s}) \mid \lambda \in \dot{\Lambda}_{r,s} \right\}$ is a basis of the center, as desired. $\square$

4. Gelfand-Zetlin subalgebras

In this section, we assume that $B_{r,s}(\delta)$ is semisimple. For the sake of simplicity, we assume further that $r \geq s$. For $1 \leq a \leq r+s$, let B_a be the subalgebra of $B_{r,s}(\delta)$ generated by

$$B_a = \begin{cases} \langle s_1, \dots, s_{a-1} \rangle & \text{if } 1 \leq a \leq r, \\ \langle s_1, \dots, s_{r-1}, e_{r,r+1}, s_{r+1} \dots s_{a-1} \rangle & \text{if } r+1 \leq a \leq r+s. \end{cases}$$

Set $B_0 := \mathbb{C}$. Then we have a tower of subalgebras

$$\mathbb{C} = B_0 \subset B_1 \subset \cdots \subset B_{r+s-1} \subset B_{r+s} = B_{r,s}(\delta).$$

This tower of subalgebras are compatible with the Jucys-Murphy elements in the following sense: for each $1 \leq a \leq r+s$, $L_1, \dots, L_a$ are contained in B_a and they are exactly the Jucys-Murphy elements of B_a under the isomorphism

$$B_a \simeq \begin{cases} B_{a,0}(\delta) & \text{if } 1 \leq a \leq r, \\ B_{r,a-r}(\delta) & \text{if } r+1 \leq a \leq r+s. \end{cases}$$

The above isomorphisms are obtained by checking that the generators of B_a produce all the (r,s)-walled Brauer diagrams with $r+s-a$ vertical strands connecting the k-th vertex on the top row with the k-th vertex on the bottom row for each $a+1 \leq k \leq r+s$. In particular, B_a is semisimple for all $0 \leq a \leq r+s$. Set

$$\Lambda_a := \begin{cases} \{\emptyset\} & \text{if } a = 0, \\ \Lambda_{a,0} & \text{if } 1 \leq a \leq r, \\ \Lambda_{r,a-r} & \text{if } r+1 \leq a \leq r+s. \end{cases}$$

For $\lambda \in \Lambda_a$, we define

$$C_a(\lambda) := \begin{cases} \mathbb{C} & \text{if } a = 0, \\ C_{a,0}(\lambda) & \text{if } 1 \leq a \leq r, \\ C_{r,a-r}(\lambda) & \text{if } r+1 \leq a \leq r+s. \end{cases}$$

For $1 \leq a \leq r$ and $\lambda \in \Lambda_a$, we have

$$\operatorname{Res}^{B_a}_{B_{a-1}} C_a(\lambda) \simeq \bigoplus_{\mu} C_{a-1}(\mu), \tag{4.1}$$

where the sum runs over the weights $\mu \in \Lambda_{a-1}$ such that the skew Young diagram $[\lambda^L]/[\mu^L]$ consists of a single box. This is nothing but the branching rule for symmetric groups. For $r+1 \leq a \leq r+s$ and $\lambda \in \Lambda_a$, it is shown in [10, Theorem 3.16] that

$$\operatorname{Res}^{B_a}_{B_{a-1}} C_a(\lambda) \simeq \bigoplus_{\mu} C_{a-1}(\mu), \tag{4.2}$$

where the sum runs over the weights $\mu \in \Lambda_{a-1}$ such that either the diagram $[\mu^L]/[\lambda^L]$ consists of a single box, or the diagram $[\lambda^R]/[\mu^R]$ consists of a single box (see also [3, Corollary 3.6]). In particular, for $1 \leq a \leq r+s$, the restriction of any simple module of B_a to B_{a-1} is multiplicity-free so that the above decompositions are canonical. From now on, we identify

$C_{a-1}(\mu)$ in the right hand side in (4.1), (4.2) with the unique simple submodule of $C_a(\lambda)$ which is isomorphic to it.

Let $\mathbf{B}$ be the *branching graph* of $B_{r,s}(\delta)$: the set of vertices is given by $\bigsqcup_{a=0}^{r+s} \Lambda_a$ and the two vertices $\mu \in \Lambda_a$ and $\lambda \in \Lambda_{a+1}$ are joined by an arrow from μ to λ if and only if $\mathrm{Hom}_{B_a}(C_a(\mu), \mathrm{Res}_{B_a}^{B_{a+1}} C_{a+1}(\lambda)) \neq 0$. Iterating the restrictions, we obtain a canonical decomposition of a simple $B_{r,s}$-module $C_{r,s}(\lambda)$ into a direct sum of simple B_0-module, i.e., 1-dimensional subspaces

$$C_{r,s}(\lambda) = \bigoplus_T V_T,$$

where the sum runs over all the paths T in $\mathbf{B}$ from $\emptyset$ to λ. Taking a non-zero vector v_T for each path T in $\mathbf{B}$ from $\emptyset$ to λ, we obtain a basis $\{v_T \mid T : \text{a path in } \mathbf{B} \text{ from } \emptyset \text{ to } \lambda\}$ of $C_{r,s}(\lambda)$, called the *Gelfand-Zetlin basis* (shortly, GZ-basis) of $C_{r,s}(\lambda)$.

Let $A_{r,s}$ be the subalgebra of $B_{r,s}(\delta)$ generated by $Z(B_1), \dots, Z(B_{r+s})$, where $Z(B_a)$ denotes the center of B_a. We call $A_{r,s}$ the *Gelfand-Zetlin subalgebra* (shortly, the GZ-subalgebra) of $B_{r,s}(\delta)$. Note that $A_{r,s}$ is commutative.

Proposition 4.1. *The GZ-subalgebra $A_{r,s}$ is generated by $L_1, \dots, L_{r+s}$.*

Proof. By Proposition 2.7, we have $L_1 + \cdots + L_{a-1} \in Z(B_{a-1})$ and $L_1 + \cdots + L_{a-1} + L_a \in Z(B_a)$. It follows that $L_a \in A_{r,s}$ for $1 \le a \le r+s$. On the other hand, by Theorem 3.5, each $Z(B_a)$ is contained in the subalgebra of B_a generated by $L_1, \dots, L_a$, and hence $A_{r,s}$ is generated by $L_1, \dots, L_{r+s}$. ☐

By Wedderbun-Artin theorem, we have a $\mathbb{C}$-algebra isomorphism

$$B_{r,s}(\delta) \simeq \bigoplus_{\lambda \in \Lambda_{r,s}} \mathrm{End}_{\mathbb{C}}(C_{r,s}(\lambda)). \tag{4.3}$$

For each $\lambda \in \Lambda_{r,s}$, we identify the algebra $\mathrm{End}_{\mathbb{C}}(C_{r,s}(\lambda))$ with the algebra of $\dim_{\mathbb{C}} C_{r,s}(\lambda) \times \dim_{\mathbb{C}} C_{r,s}(\lambda)$ matrices over $\mathbb{C}$, by taking a GZ-basis of $C_{r,s}(\lambda)$. Then we have the following proposition whose proof is identical to the one of [28, Proposition 1.1].

Proposition 4.2. *The GZ-subalgebra $A_{r,s}$ is identified with the set of all the diagonal matrices. In particular, $A_{r,s}$ is a maximal commutative subalgebra of $B_{r,s}(\delta)$.*

For a path T in $\mathbf{B}$ given by

$$T = \emptyset \to \lambda_1 \to \cdots \to \lambda_{r+s-1} \to \lambda_{r+s}, \quad (\lambda_a \in \Lambda_a),$$

set

$$c_T(i) := \begin{cases} \text{content of } [\lambda_i^L]/[\lambda_{i-1}^L] & \text{if } 1 \le i \le r, \\ -\text{content of } [\lambda_{i-1}^L]/[\lambda_i^L] & \text{if } r+1 \le i \le r+s,\ [\lambda_{i-1}^L]/[\lambda_i^L] \text{ is a single box} \\ \text{content of } [\lambda_i^R]/[\lambda_{i-1}^R] + \delta, & \text{if } r+1 \le i \le r+s,\ [\lambda_i^R]/[\lambda_{i-1}^R] \text{ is a single box.} \end{cases}$$

The below was shown for the case $1 \le i \le r$ in [24] (see also [5, Theorem 1.1] and [28, Section 5]).

Proposition 4.3. *Let T be given as the above. Then we have*

$$L_i\, v_T = c_T(i) v_T \quad (1 \leq i \leq r+s).$$

Proof. For each $\lambda \in \bigsqcup_{a=0}^{r+s} \Lambda_a$, set

$$c(\lambda) := \sum_{j=1}^{|\lambda^L|} \mathrm{cont}(\lambda^L, j) + \sum_{j=1}^{|\lambda^R|} (\mathrm{cont}(\lambda^R, j) + \delta).$$

Observe that

$$c(\lambda_i) - c(\lambda_{i-1}) = c_T(i) \quad \text{for} \ \ 1 \leq i \leq r+s.$$

On the other hand, by Proposition 3.1, we know

$$(L_1 + \cdots + L_i) v_T = c(\lambda_i) v_T \quad \text{for} \ \ 1 \leq i \leq r+s.$$

In particular, we have

$$L_i v_T = (c(\lambda_i) - c(\lambda_{i-1})) v_T = c_T(i) v_T,$$

as desired. □

The following is an analogue of [25, Theorem 2.1]. See also [21, Definition 3.1].

Proposition 4.4. *For each $1 \leq i \leq r+s$, set*

$$\mathcal{C}(i) := \{ c_T(i) \mid \ T : \textit{a path in } \mathbf{B} \textit{ from } \emptyset \textit{ to } \lambda \textit{ for some } \lambda \in \Lambda_{r+s} \}.$$

The following elements form a complete set of primitive orthogonal idempotents of $B_{r,s}(\delta)$:

$$I_T := \prod_{i=1}^{r+s} \prod_{\substack{c \in \mathcal{C}(i) \\ c \neq c_T(i)}} \frac{L_i - c}{c_T(i) - c}.$$

Proof. Combining Proposition 4.1 and Proposition 4.2, we know that $T = T'$ if and only if $c_T(i) = c_{T'}(i)$ for all $1 \leq i \leq r+s$ (see [28, Remark 1.2]). Hence we have

$$I_T v_{T'} = \prod_{i=1}^{r+s} \prod_{\substack{c \in \mathcal{C}(i) \\ c \neq c_T(i)}} \frac{c_{T'}(i) - c}{c_T(i) - c} = \delta_{T,T'} v_{T'},$$

as desired. □

Remark 4.5. The above can be proved by checking that a path T in $\mathbf{B}$ is uniquely determined by $(c_T(i))_{1 \leq i \leq r+s}$, whenever the triple (r, s, δ) belongs to one of the cases in Proposition 1.1.

5. Blocks of walled Brauer algebra

In this section, we return to the cases in which r, s and δ are arbitrarily chosen. We say that two simple $B_{r,s}(\delta)$-modules $D_{r,s}(\lambda)$ and $D_{r,s}(\mu)$ *belong to the same block* if there exists a sequence of simple $B_{r,s}(\delta)$-modules $D_{r,s}(\lambda) = D_1, D_2, ..., D_k = D_{r,s}(\mu)$ such that either $\mathrm{Ext}^1_{B_{r,s}(\delta)}(D_i, D_{i+1}) \neq 0$ or $\mathrm{Ext}^1_{B_{r,s}(\delta)}(D_{i+1}, D_i) \neq 0$, for all $1 \leq i < k$. For each $\lambda \in \dot{\Lambda}_{r,s}$, every element z in the center acts on $D_{r,s}(\lambda)$ by a scalar multiple, say $\psi_\lambda(z)$. The assignment $z \mapsto \psi_\lambda(z)$ defines a $\mathbb{C}$-algebra homomorphism

$$\psi_\lambda : Z(B_{r,s}(\delta)) \to \mathbb{C},$$

and we call ψ_λ the *central character afforded by* $D_{r,s}(\lambda)$. Two simple modules $D_{r,s}(\lambda)$ and $D_{r,s}(\mu)$ belong to the same block if and only if the central characters ψ_λ and ψ_μ are identical (see, for example, [13, Chapter I, Proposition 10.15]). Note that the scalar multiplication on $C_{r,s}(\lambda)$ induced by a central element z equals to $\psi_\lambda(z)$, because $D_{r,s}(\lambda)$ is a quotient of $C_{r,s}(\lambda)$.

We will say that (λ^L, λ^R) and (μ^L, μ^R) are *δ-balanced* if there is a pairing of the boxes in $[\lambda^L]/([\lambda^L] \cap [\mu^L])$ with those in $[\lambda^R]/([\lambda^R] \cap [\mu^R])$ and a pairing of the boxes in $[\mu^L]/([\lambda^L] \cap [\mu^L])$ with those in $[\mu^R]/([\lambda^R] \cap [\mu^R])$ such that the contents of each pair sum to $-\delta$. In [3], the blocks of $B_{r,s}(\delta)$ are classified as follows:

Proposition 5.1. ([3, Corollary 7.7]) *Two simple modules $D_{r,s}(\lambda)$ and $D_{r,s}(\mu)$ are in the same block of $B_{r,s}(\delta)$ if and only if the weights λ and μ are δ-balanced.*

The following lemma shows that the notion of δ-balanced weights is closely related to the contents evaluation of supersymmetric polynomials.

Lemma 5.2. *Two weights $\lambda = (\lambda^L, \lambda^R)$ and $\mu = (\mu^L, \mu^R)$ are* δ-balanced *if and only if*

$$p((c(\lambda, i))_{1 \leq i \leq r+s}) = p((c(\mu, i))_{1 \leq i \leq r+s})$$

for every supersymmetric polynomial $p \in S_{r,s}[x; y]$.

Proof. Recall that for $\lambda, \mu \in \Lambda_{r,s}$ we have

$$p((c(\lambda, i))_{1 \leq i \leq r+s}) = p((c(\mu, i))_{1 \leq i \leq r+s})$$

for every supersymmetric polynomial $p \in S_{r,s}[x; y]$ if and only if the equation (2.1)

$$\frac{\prod_{i=1}^{r-t}(1 + \mathrm{cont}(\lambda^L, i)z)}{\prod_{j=r+t+1}^{r+s}(1 - (\mathrm{cont}(\lambda^R, j) + \delta)z)} = \frac{\prod_{i=1}^{r-t'}(1 + \mathrm{cont}(\mu^L, i)z)}{\prod_{j=r+t'+1}^{r+s}(1 - (\mathrm{cont}(\mu^R, j) + \delta)z)}$$

holds. It is equivalent to saying that

$$\begin{aligned} & \sharp\left\{u \in [\lambda^L] \mid \text{content of the box } u = k\right\} - \sharp\left\{u \in [\lambda^R] \mid \text{content of the box } u = -\delta - k\right\} \\ = \quad & \sharp\left\{u \in [\mu^L] \mid \text{content of the box } u = k\right\} - \sharp\left\{u \in [\mu^R] \mid \text{content of the box } u = -\delta - k\right\} \end{aligned}$$

for all $k \in \mathbb{Z}$.

It is shown in [3, Lemma 8.1] that the above condition is equivalent to saying that λ and μ are δ-balanced. □

We give an alternative proof of a part of the classification theorem of blocks of $B_{r,s}(\delta)$, appeared in [3].

Proposition 5.3. ([3, Corollary 7.3]) *If two simple modules $D_{r,s}(\lambda)$ and $D_{r,s}(\mu)$ are in the same block of $B_{r,s}(\delta)$, then the weights λ and μ are δ-balanced.*

Proof. We have $\psi_\lambda = \psi_\mu$. In particular, by Corollary 2.9, and Corollary 3.3, we have

$$p((c(\lambda,i))_{1\le i\le r+s}) = \psi_\lambda(p(L_1,\dots,L_{r+s})) = \psi_\mu(p(L_1,\dots,L_{r+s})) = p((c(\mu,i))_{1\le i\le r+s})$$

for every supersymmetric polynomial $p \in S_{r,s}[x;y]$. Hence λ and μ are δ-balanced by the above lemma. □

We expect that the following generalization of our main theorem holds. Note that it corresponds to [32, Conjecture 7.4], if we take the modification given in Remark 2.2 and focus on $B_{r,s}(\delta)$.

Conjecture 5.4. (see also [32, Conjecture 7.4]) *For every $\delta \in \mathbb{C}$, the center of walled Brauer algebra $B_{r,s}(\delta)$ is generated by the supersymmetric polynomials in the Jucys-Murphy elements $L_1,\dots,L_{r+s}$.*

Remark 5.5. If the above conjecture is true, then the central characters ψ_λ and ψ_μ are identical for every δ-balanced pair λ and μ. Indeed, the central characters are given by the contents evaluation of supersymmetric polynomials in Jucys-Murphy elements, and by Lemma 5.2, the contents evaluations are identical whenever λ and μ are δ-balanced. In turn, the simple modules $D_{r,s}(\lambda)$ and $D_{r,s}(\mu)$ belong to the same block. It would recover [3, Corollary 7.7], the other direction of the classification of the blocks of $B_{r,s}(\delta)$.

We close the paper with an example supporting the above conjecture in small degree.

Example 5.6. Let $r = 2, s = 2$. By the method in [35], we obtain a basis of the centralizer $Z_{B_{2,2}(\delta)}(\mathbb{C}[\mathfrak{S}_2 \times \mathfrak{S}_2])$ of the subalgebra $\mathbb{C}[\mathfrak{S}_2 \times \mathfrak{S}_2]$ in $B_{2,2}(\delta)$. Each element of the basis corresponds to a *walled generalized cycle types*. For the detail, see [35, Section 7]. We enumerate thus obtained elements as follows:

$$\begin{aligned}
&C_1 = \mathbf{1}, \quad C_2 = (1,2), \quad C_3 = (3,4) \quad C_4 = e_{2,3} + e_{1,3} + e_{1,4} + e_{2,4},\\
&C_5 = e_{1,3}e_{2,4} + e_{1,4}e_{2,3}, \quad C_6 = C_2C_3, \quad C_7 = C_2C_4, \quad C_8 = C_3C_4,\\
&C_9 = C_2C_5, \quad C_{10} = C_4C_6.
\end{aligned}$$

In particular, we have $\dim Z_{B_{2,2}(\delta)}(\mathbb{C}[\mathfrak{S}_2 \times \mathfrak{S}_2]) = 10$. Note that an element in the centralizer $Z_{B_{2,2}(\delta)}(\mathbb{C}[\mathfrak{S}_2 \times \mathfrak{S}_2])$ is central in $B_{2,2}(\delta)$ if and only if it commutes with the generator $e_{2,3}$. Now the equation

$$\Big(\sum_{i=1}^{10} a_iC_i\Big)e_{2,3} - e_{2,3}\Big(\sum_{i=1}^{10} a_iC_i\Big) = 0$$

yields the following system of 4 linear equations:

$$\begin{aligned}
&a_2 + a_4 + \delta a_7 + a_{10} = 0, \quad a_3 + a_4 + \delta a_8 + a_{10} = 0,\\
&a_5 + a_7 + a_8 + \delta a_9 = 0, \quad a_6 + a_7 + a_8 + \delta a_{10} = 0.
\end{aligned}$$

From the above, we obtain a basis of $Z(B_{2,2}(\delta))$:

$$\{B_1 := \mathbf{1},\ B_2 := C_4 - C_2 - C_3,\ B_3 := C_8 - \delta C_3 - C_5 - C_6,\ B_4 := C_7 - \delta C_2 - C_5 - C_6,\\ B_5 := C_9 - \delta C_5,\ B_6 = C_{10} - C_3 - \delta C_6 - C_2\ \}.$$

In particular, we have $\dim Z(B_{2,2}(\delta)) = 6$, which is independent from δ (see also the conjecture on the dimension of the center of the Brauer algebra in [35, Introduction]).

On the other hand, we obtain the following equalities by direct calculations:

$$\begin{aligned}
p_0(L_1, L_2, L_3, L_4) \quad &= \mathbf{1} = B_1,\\
p_1(L_1, L_2, L_3, L_4) \quad &= 2\delta C_1 + C_2 + C_3 - C_4 = 2\delta B_1 - B_2,\\
e_2(L_1, L_2, L_3, L_4) \quad &= (3\delta^2+1)C_1 + 2\delta C_2 + 3\delta C_3 - 2\delta C_4 + C_5 + C_6 - C_8\\
&= (3\delta^2+1)B_1 - 2\delta B_2 - B_3,\\
p_2(L_1, L_2, L_3, L_4) \quad &= -2\delta^2 C_1 - 2\delta C_3 + \delta C_4 - C_7 + C_8 = -2\delta^2 B_1 + \delta B_2 + B_3 - B_4,\\
e_3(L_1, L_2, L_3, L_4) \quad &= (4\delta^3+4\delta)C_1 + (3\delta^2+1)C_2 + (6\delta^2+1)C_3 + (-3\delta^2-1)C_4 + 2\delta C_5\\
&\quad +3\delta C_6 - 3\delta C_8 + C_9\\
&= (4\delta^3+4\delta)B_1 + (-3\delta^2-1)B_2 - 3\delta B_3 + B_5,\\
p_3(L_1, L_2, L_3, L_4) \quad &= (2\delta^3+3\delta)C_1 + C_2 + (3\delta^2+1)C_3 + (-\delta^2-2)C_4 + \delta C_7 - 2\delta C_8 + C_9 + C_{10}\\
&= (2\delta^3+3\delta)B_1 + (-\delta^2-2)B_2 - 2\delta B_3 + \delta B_4 + B_5 + B_6.
\end{aligned}$$

Observe that the matrix

$$\begin{pmatrix}
1 & 2\delta & 3\delta^2+1 & -2\delta^2 & 4\delta^3+4\delta & 2\delta^3+3\delta\\
0 & -1 & -2\delta & \delta & -3\delta^2-1 & -\delta^2-2\\
0 & 0 & -1 & 1 & -3\delta & -2\delta\\
0 & 0 & 0 & -1 & 0 & \delta\\
0 & 0 & 0 & 0 & 1 & 1\\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

is invertible for every $\delta \in \mathbb{C}$. Hence the evaluation of $\{p_0, p_1, p_2, p_3, e_2, e_3\}$ at L_1, L_2, L_3, L_4 becomes a basis of the center $Z(B_{2,2}(\delta))$ for every $\delta \in \mathbb{C}$, and hence the supersymmetric polynomials in L_1, L_2, L_3, L_4 generate the center.

6. Center of the quantized walled Brauer algebra

In this section, we establish an analogue of Theorem 3.5 for the quantized walled Brauer algebra $H_{r,s}(q,\rho)$. The following definition appeared in [15, 19].

Definition 6.1. Let r and s be nonnegative integers. Let R be an integral domain and let q, ρ be elements in R such that q^{-1}, ρ^{-1} and $\delta := \frac{\rho-\rho^{-1}}{q-q^{-1}}$ lie in R. We denote by $H^R_{r,s}(q,\rho)$ the associative algebra with $\mathbf{1}$ over R generated by $S_1, \ldots, S_{r-1}$, $S_{r+1}, \ldots,$ S_{r+s-1}, $E_{r,r+1}$ with

the defining relations

$$\begin{aligned}
&(S_i - q)(S_i + q^{-1}) = 0, \quad S_i S_{i+1} S_i = S_{i+1} S_i S_{i+1}, \quad S_i S_j = S_j S_i \ \ (|i-j| > 1),\\
&E_{r,r+1}^2 = \delta E_{r,r+1}, \quad E_{r,r+1} S_j = S_j E_{r,r+1} \ \ (j \neq r-1, r+1),\\
&\rho E_{r,r+1} = E_{r,r+1} S_{r-1} E_{r,r+1} = E_{r,r+1} S_{r+1} E_{r,r+1},\\
&E_{r,r+1} S_{r-1}^{-1} S_{r+1} E_{r,r+1} S_{r-1} = E_{r,r+1} S_{r-1}^{-1} S_{r+1} E_{r,r+1} S_{r+1},\\
&S_{r-1} E_{r,r+1} S_{r-1}^{-1} S_{r+1} E_{r,r+1} = S_{r+1} E_{r,r+1} S_{r-1}^{-1} S_{r+1} E_{r,r+1}.
\end{aligned}$$

Note that the element S_i^{-1} in the last two relations is given by $S_i^{-1} = S_i - (q - q^{-1})\mathbf{1}$ from the relation in the first line.

By [19], the algebra $H_{r,s}^R(q,\rho)$ has an R-linear basis which is in bijection with (r,s)-walled Brauer diagrams: for each (r,s)-walled Brauer diagram c, we attach a monomial $T_c \in H_{r,s}^R(q,\rho)$ in $S_i, S_i^{-1}, E_{r,r+1}$ so that the expression of each basis element does not depend on the choice of the base ring R. See [19] for the explicit expression. It is called the *standard monomial basis*. See also [6, 10, 16].

Now assume that we have a ring homomorphism $\phi : R \to S$ for some integral domain S. Then $H_{r,s}^S(\phi(q),\phi(\rho))$ becomes an R-algebra via ϕ, and we have an R-algebra homomorphism $\tilde{\phi} : H_{r,s}^R(q,\rho) \to H_{r,s}^S(\phi(q),\phi(\rho))$, sending the generators $E_{r,r+1}$ and S_i's to themselves. By the definition of standard monomial basis, the homomorphism $\widetilde{\phi}$ sends T_c to T_c for each (r,s)-walled Brauer diagram c. Note that $\{T_c\} \subset H_{r,s}^S(\phi(q),\phi(\rho))$ is a linearly independent subset over R if and only if ϕ is injective. Hence, if ϕ is injective, then $\tilde{\phi}$ is also injective and the image of $\widetilde{\phi}$ is isomorphic to $H_{r,s}^R(q,\rho)$ as an R-algebra. In particular, the image of $\widetilde{\phi}$ has an R-basis $\{T_c\}$ so that it can be characterized as the R-subalgebra of $H_{r,s}^S(\phi(q),\phi(\rho))$ generated by $E_{r,r+1}, S_i$ $(i = 1, \dots, r-1, r+1, \dots, r+s-1)$.

The following definition is motivated by Definition 2.1 as well as [27].

Definition 6.2. Set

$$\mathcal{L}_1 := 0, \quad \mathcal{L}_{r+1} := \rho\Big(\sum_{j=1}^{r} -E_{j,r+1} + \delta\Big) \ \in \ H_{r,s}^R(q,\rho),$$

where

$$E_{j,k} := (S_{k-1}\cdots S_{r+1})(S_j^{-1}\cdots S_{r-1}^{-1})E_{r,r+1}(S_{r-1}^{-1}\cdots S_j^{-1})(S_{r+1}\cdots S_{k-1}) \quad \text{for} \ \ j \le r < k.$$

Then we define

$$\mathcal{L}_k := \begin{cases} S_{k-1}^{-1}\mathcal{L}_{k-1}S_{k-1}^{-1} + S_{k-1}^{-1} & \text{if} \ \ 2 \le k \le r,\\ S_{k-1}\mathcal{L}_{k-1}S_{k-1} + S_{k-1} & \text{if} \ \ r+2 \le k \le r+s. \end{cases} \tag{6.1}$$

We call these $\mathcal{L}_k$'s the *Jucys-Murphy elements of* $H_{r,s}^R(q,\rho)$.

For convenience, we set

$$T_{(a,b)} = T_{(b,a)} := \begin{cases} S_{b-1}^{-1}\cdots S_{a+1}^{-1}S_a^{-1}S_{a+1}^{-1}\cdots S_{b-1}^{-1} & \text{for } 1 \le a < b \le r,\\ S_{b-1}\cdots S_{a+1}S_aS_{a+1}\cdots S_{b-1} & \text{for } r+1 \le a < b \le r+s. \end{cases}$$

Calculating $\mathcal{L}_k$, we have

$$\mathcal{L}_k = \begin{cases} \sum_{j=1}^{k-1} T_{(j,k)} & \text{if } \ 2 \leq k \leq r, \\ \rho\big(\sum_{j=1}^{r} -E_{j,k} + \delta\big) + \rho^2 \sum_{j=r+1}^{k-1} T_{(j,k)} & \text{if } \ r+2 \leq k \leq r+s. \end{cases}$$

Remark 6.3. The elements $\mathcal{L}_1, \ldots, \mathcal{L}_r$ can be considered as Jucys-Murphy elements of Hecke algebra $H_r^R(q)$, which were firstly introduced in [7]. Indeed, if we replace $T_{(i,i+1)}$ and q in [7] with $q^{-1}S_i^{-1}$ and q^{-2}, respectively, we get $\widetilde{L}_i = q\mathcal{L}_i$ $(i \geq 2)$, where $\widetilde{L}_i$ is the image of *Murphy operator* called in [7].

Proposition 6.5 is analogous to Proposition 2.4. To prove it, we need a lemma.

Lemma 6.4. *For all admissible i, j, k, we have*
(1) $S_iT_{(j,k)} = T_{(j,k)}S_i$ *and* $S_iE_{j,k} = E_{j,k}S_i$ *for* $i, i+1 \neq j, k$,
(2) $S_iE_{i,k} = E_{i+1,k}S_i - (q-q^{-1})E_{i+1,k}$ *and* $E_{i,k}S_i = S_iE_{i+1,k} - (q-q^{-1})E_{i+1,k}$,
(3) $E_{r,r+1}E_{j,k} = E_{j,k}E_{r,r+1}$ *for* $j \neq r, k \neq r+1$,
(4) $E_{r,r+1}E_{i,r+1} = \rho^{-1}E_{r,r+1}T_{(i,r)}$ *and* $E_{i,r+1}E_{r,r+1} = \rho^{-1}T_{(i,r)}E_{r,r+1}$ *for* $i \neq r$,
$E_{r,r+1}E_{r,k} = \rho\, E_{r,r+1}T_{(r+1,k)}$ *and* $E_{r,k}E_{r,r+1} = \rho\, T_{(r+1,k)}E_{r,r+1}$ *for* $k \neq r+1$.

Proof. (1)The first relation can be checked from the defining relations $S_iS_{i+1}S_i = S_{i+1}S_iS_{i+1}$ and $S_iS_j = S_jS_i$ $(|i-j| > 1)$. For the second relation, it is trivial when $i \leq j-2$ or $k+1 \leq i$. Let $j+1 \leq i < r$. Note that $S_{i-1}E_{r,k} = E_{r,k}S_{i-1}$. We have

$$\begin{aligned} S_i^{-1}E_{j,k} &= S_i^{-1}S_j^{-1}\cdots S_{r-1}^{-1}E_{r,k}S_{r-1}^{-1}\cdots S_j^{-1} \\ &= S_j^{-1}\cdots S_i^{-1}S_{i-1}^{-1}S_i^{-1}\cdots S_{r-1}^{-1}E_{r,k}S_{r-1}^{-1}\cdots S_j^{-1} \\ &= S_j^{-1}\cdots S_{i-1}^{-1}S_i^{-1}S_{i-1}^{-1}\cdots S_{r-1}^{-1}E_{r,k}S_{r-1}^{-1}\cdots S_j^{-1} \\ &= S_j^{-1}\cdots S_{r-1}^{-1}E_{r,k}S_{r-1}^{-1}\cdots S_{i-1}^{-1}S_i^{-1}S_{i-1}^{-1}\cdots S_j^{-1} \\ &= S_j^{-1}\cdots S_{r-1}^{-1}E_{r,k}S_{r-1}^{-1}\cdots S_i^{-1}S_{i-1}^{-1}S_i^{-1}\cdots S_j^{-1} \\ &= S_j^{-1}\cdots S_{r-1}^{-1}E_{r,k}S_{r-1}^{-1}\cdots S_j^{-1}S_i^{-1} \\ &= E_{j,k}S_i^{-1}, \end{aligned}$$

hence $E_{j,k}S_i = S_iE_{j,k}$. Similarly, we can obtain $S_iE_{j,k} = E_{j,k}S_i$ for $r < i \leq k-2$.

(2) By the definition of $E_{i,k}$, we get

$$S_iE_{i,k} = E_{i+1,k}S_i^{-1} = E_{i+1,k}(S_i - (q-q^{-1})\mathbf{1}) = E_{i+1,k}S_i - (q-q^{-1})E_{i+1,k}.$$

Similarly, we can obtain another relation.

(3) From the last two defining relations of $H_{r,s}^R(q,\rho)$, we get

$$E_{r,r+1}S_{r-1}^{-1}S_{r+1}E_{r,r+1}S_{r+1}S_{r-1}^{-1} = S_{r-1}^{-1}S_{r+1}E_{r,r+1}S_{r-1}^{-1}S_{r+1}E_{r,r+1},$$

that is, $E_{r,r+1}E_{r-1,r+2} = E_{r-1,r+2}E_{r,r+1}$. Using it and relations $S_iE_{r,r+1} = E_{r,r+1}S_i$ $(i \neq r-1, r+1)$, we get the desired relations.

(4) From the defining relations of $H_{r,s}^R(q,\rho)$, we have $\rho^{-1}E_{r,r+1} = E_{r,r+1}S_{r-1}^{-1}E_{r,r+1}$. Using it and relations $S_iE_{r,r+1} = E_{r,r+1}S_i$ $(i \neq r-1, r+1)$, we obtain the first relation as follows:

$$E_{r,r+1}E_{i,r+1} = E_{r,r+1}S_i^{-1}\cdots S_{r-1}^{-1}E_{r,r+1}S_{r-1}^{-1}\cdots S_i^{-1}$$

$$
\begin{aligned}
&= S_i^{-1} \cdots S_{r-2}^{-1} E_{r,r+1} S_{r-1}^{-1} E_{r,r+1} S_{r-1}^{-1} \cdots S_i^{-1} \\
&= \rho^{-1} S_i^{-1} \cdots S_{r-2}^{-1} E_{r,r+1} S_{r-1}^{-1} \cdots S_i^{-1} \\
&= \rho^{-1} E_{r,r+1} T_{(i,r)}.
\end{aligned}
$$

Similarly we can get the second relation. Using $\rho E_{r,r+1} = E_{r,r+1} S_{r+1} E_{r,r+1}$, we obtain the third and the fourth relation. □

Proposition 6.5. *We have the following relations:*

(1) $S_i \mathcal{L}_{i+1} - \mathcal{L}_i S_i = 1 - (q - q^{-1})\mathcal{L}_i$ *for* $i < r$,
$\mathcal{L}_{i+1} S_i - S_i \mathcal{L}_i = 1 - (q - q^{-1})\mathcal{L}_i$ *for* $i < r$,
(2) $S_i \mathcal{L}_{i+1} - \mathcal{L}_i S_i = 1 + (q - q^{-1})\mathcal{L}_{i+1}$ *for* $i > r$,
$\mathcal{L}_{i+1} S_i - S_i \mathcal{L}_i = 1 + (q - q^{-1})\mathcal{L}_{i+1}$ *for* $i > r$,
(3) $E_{r,r+1}(\mathcal{L}_r + \mathcal{L}_{r+1}) = (\mathcal{L}_r + \mathcal{L}_{r+1})E_{r,r+1} = 0$,
(4) $S_i \mathcal{L}_k = \mathcal{L}_k S_i$ *for* $k \neq i, i+1$,
(5) $E_{r,r+1} \mathcal{L}_k = \mathcal{L}_k E_{r,r+1}$ *for* $k \neq r, r+1$.

Proof. (1),(2) The relations can be checked from the definition of $\mathcal{L}_i$ in (6.1) and the defining relations $(S_i - q)(S_i + q^{-1}) = 0$. The relations in (1) can be also obtained from [7, Lemma 2.3(ii),(iii)].

(3) From Lemma 6.4 (4) and $E_{r,r+1}^2 = \delta E_{r,r+1}$, we get

$$
\begin{aligned}
E_{r,r+1}(\mathcal{L}_r + \mathcal{L}_{r+1}) \qquad &= E_{r,r+1}\big(\sum_{j=1}^{r-1} T_{(j,r)} + \rho \sum_{j=1}^{r} -E_{j,r+1} + \rho\delta\big) \\
&= \sum_{j=1}^{r-1} E_{r,r+1} T_{(j,r)} - E_{r,r+1} T_{(j,r)} - \rho\delta E_{r,r+1} + \rho\delta E_{r,r+1} = 0.
\end{aligned}
$$

By similar manner, we can obtain the second relation.

(4) The case $i \geq k+1$ is trivial. Let $i < k-1 < k \leq r$. Then one can check that

$$
S_i(T_{(i,k)} + T_{(i+1,k)}) = (T_{(i,k)} + T_{(i+1,k)})S_i. \tag{6.2}
$$

Here, we use the relations

$$
S_i T_{(i,k)} = T_{(i+1,k)} S_i - (q - q^{-1}) T_{(i+1,k)}, \ T_{(i,k)} S_i = S_i T_{(i+1,k)} - (q - q^{-1}) T_{(i+1,k)},
$$

which can be obtained from the defining relations of $H_{r,s}^R(q, \rho)$. Using (6.2) and Lemma 6.4 (1), we obtain $S_i \mathcal{L}_k = \mathcal{L}_k S_i$.

Let $i < r < k$. By Lemma 6.4 (2), we have $S_i(E_{i,k} + E_{i+1,k}) = (E_{i,k} + E_{i+1,k})S_i$. From it and Lemma 6.4 (1), we obtain the desired result. For $r < i < k-1$, we can get the same relation as (6.2) from the relations

$$
S_i T_{(i,k)} = T_{(i+1,k)} S_i + (q - q^{-1}) T_{(i,k)}, \ T_{(i,k)} S_i = S_i T_{(i+1,k)} + (q - q^{-1}) T_{(i,k)}.
$$

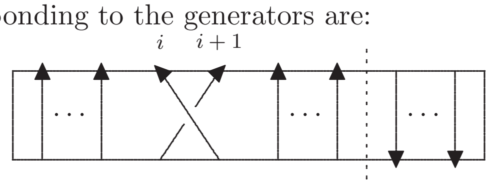
From it and Lemma 6.4 (1), we can obtain $S_i \mathcal{L}_k = \mathcal{L}_k S_i$, as desired.

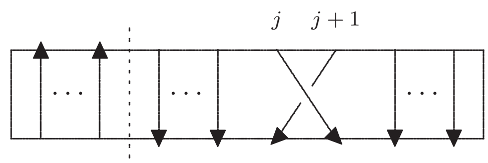
(5) If $k < r$, it is trivial. Let $k \geq r+2$. From Lemma 6.4 (1),(3) and (4), we can obtain $E_{r,r+1} \mathcal{L}_k = E_{r,r+1} \mathcal{L}_k$. □

Corollary 6.6. *The elements $\mathcal{L}_k$'s are commuting with each other and every supersymmetric polynomials in $\mathcal{L}_1, \dots, \mathcal{L}_{r+s}$ belong to the center $Z(H^R_{r,s}(q,\rho))$ of $H^R_{r,s}(q,\rho)$.*

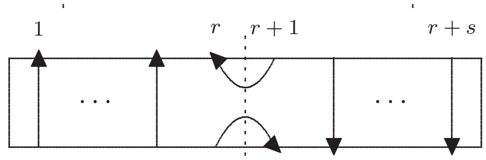
Proof. Replacing $(i, i+1) = s_i, e_{r,r+1}$ with $S_i, E_{r,r+1}$, respectively, we can obtain the results by the same proofs as for Proposition 2.6 and 2.7. $\square$

Remark 6.7. (1) The fact that $\mathcal{L}_1 + \cdots + \mathcal{L}_{r+s}$ belongs to $Z(H^R_{r,s}(q,\rho))$ was also shown in [30, Proposition 2.5]. Indeed, the element $c_{r,s}$ in [30, Proposition 2.5] is $-\rho^{-1}(\mathcal{L}_1 + \cdots + \mathcal{L}_{r+s}) + s\delta$.
(2) Set $\Lambda := \mathbb{Z}[q^{\pm 1}, \rho^{\pm 1}]_{(q-q^{-1})}$, the localization of the Laurent polynomial ring in q and ρ at $(q - q^{-1})$. Then the above corollary was shown in [27, Theorem 1] by a skein theoretic description of $H^\Lambda_{r,s}(q,\rho)$. Let us explain it more precisely. Hugh Morton studied a relation between the framed HOMFLY skein module on the annulus and the center of the framed HOMFLY skein module on the rectangle with designated inputs and outputs boundary points. The latter skein module with a naturally defined multiplication is isomorphic to $H^\Lambda_{r,s}(q,\rho)$, where the parameters s and v used there correspond to q^{-1} and ρ, respectively. The diagrams corresponding to the generators are:

$S_i =$ 

, $S_j =$ 

,

$E_{r,r+1} =$ 

.

Morton introduced certain elements $T(j), U(k)$, which have the relations with $\mathcal{L}_i$'s as follows:

$$T(j) = -(q - q^{-1})\mathcal{L}_j + 1 \quad (1 \le j \le r),$$
$$U(k) = \rho^{-2}(q - q^{-1})\mathcal{L}_{r+k} + \rho^{-2} \quad (1 \le k \le s).$$

The elements $T(j)$ and $U(k)$, which is called *Murphy operators* in [27], correspond to simple braid diagrams in the skein theoretic description. See [27, Section 3] for the diagrammatic presentation of $T(j)$ and $U(k)$. It was shown that the elements of the form

$$\sum_{j=1}^{r} (\rho^{-1} T(j))^m - \sum_{k=1}^{s} (\rho U(k))^m \quad (m \in \mathbb{Z}_{\ge 0})$$

belong to the center of $H^\Lambda_{r,s}(q,\rho)$. Moreover it was conjectured that the above elements generate the center of $H^\Lambda_{r,s}(q,\rho)$. Note that it is equivalent to saying that the supersymmetric polynomials in $\mathcal{L}_i$'s generate the center of $H^\Lambda_{r,s}(q,\rho)$.

(3) Recently A. M. Semikhatov and I. Y. Tipunin introduced some elements $\{J(r)_i\,;\, 1 \le i \le r+s\}$ having similar diagrammatic presentations to $T(j)$ and $U(k)$, which are also called *Jucys-Murphy elements* in [33, Section 2.4]. In particular, $J(r)_{r+k}$ coincides with $U(k)$ for $1 \le k \le s$ under an isomorphism between their algebra $\mathsf{qw}\mathcal{B}_{r,s}$ and $H^{\mathbb{C}(q,\rho)}_{r,s}(q,\rho)$.

We set $H_{r,s}(q,\rho) := H_{r,s}^{\mathbb{C}(q,\rho)}(q,\rho)$ and $H_{r,s}(N) := H_{r,s}^{\mathbb{C}(q)}(q,q^N)$ for $N \in \mathbb{Z}_{\geq}0$, respectively. The latter algebra appeared in [16] for the first time. Note that $H_{r,s}^{\mathbb{C}(q,\rho)}(q,\rho)$ is semisimple by, for example, [30, Theorem 6.10]. We call these two families the *quantized walled Brauer algebras*. For the representation theory of these algebras, see, for example, [6, 8, 30] and references therein. In particular, the set $\dot{\Lambda}_{r,s}$ parametrizes the isomorphism classes of simple modules of $H_{r,s}(q,\rho)$ and $H_{r,s}(N)$ as well as the ones of $B_{r,s}(\delta)$. Thus we know that

$$\dim_{\mathbb{C}(q,\rho)} Z(H_{r,s}(q,\rho)) = \dim_{\mathbb{C}} Z(B_{r,s}(\delta)),$$

where δ is generic. If $H_{r,s}(N)$ are semisimple, then

$$\dim_{\mathbb{C}} Z(B_{r,s}(N)) = \dim_{\mathbb{C}(q)} Z(H_{r,s}(N)),$$

since $B_{r,s}(N)$ is also semisimple by comparing [30, Theorem 6.10] with Proposition 1.1.

Theorem 6.8. *Suppose that $H_{r,s}(N)$ are semisimple. Then the center $Z(H_{r,s}(N))$ of $H_{r,s}(N)$ is generated by the supersymmetric polynomials in the Jucys-Murphy elements $\mathcal{L}_1,\dots,\mathcal{L}_{r+s}$ with coefficients in $\mathbb{C}(q)$.*

The center $Z(H_{r,s}(q,\rho))$ of $H_{r,s}(q,\rho)$ is also generated by the supersymmetric polynomials in the Jucys-Murphy elements $\mathcal{L}_1,\dots,\mathcal{L}_{r+s}$ with coefficients in $\mathbb{C}(q,\rho)$.

Proof. Set $d = \dim Z(B_{r,s}(N)) = |\dot{\Lambda}_{r,s}|$. Let

$$\{P_i \in S_{r,s}[x;y] \mid 1 \leq i \leq d\}$$

be a set of supersymmetric polynomials such that $\{P_i(L_1,\dots,L_{r+s}) \in B_{r,s}(N) \mid 1 \leq i \leq d\}$ forms a basis of $Z(B_{r,s}(N))$. We will show first that

$$\{P_i(\mathcal{L}_1,\dots,\mathcal{L}_{r+s}) \in H_{r,s}(N) \mid 1 \leq i \leq d\}$$

is a linearly independent subset of $H_{r,s}(N)$ over $\mathbb{C}(q)$. Assume that there exists a nontrival $\mathbb{C}(q)$-linear relation

$$\sum_{i=1}^{d} a_i(q) P_i(\mathcal{L}_1,\dots,\mathcal{L}_{r+s}) = 0.$$

Then by multiplying the least common multiple of the denominators, we may assume that $a_i(q) \in \mathbb{C}[q]$ for all i. Dividing the above by a suitable power of $q-1$, we may further assume that there exists i_0 such that $a_{i_0}(1) \neq 0$. Now let us denote by $H_{\mathbb{C}[q^{\pm 1}]}$ the $\mathbb{C}[q^{\pm 1}]$-subalgebra of $H_{r,s}(N)$ generated by $E_{r,r+1}$ and S_i's. Recall that $H_{\mathbb{C}[q^{\pm 1}]}$ is isomorphic to $H_{r,s}^{\mathbb{C}[q^{\pm 1}]}(q,q^N)$. Then we have a ring homomorphism $\phi_{q=1} : \mathbb{C}[q^{\pm 1}] \to \mathbb{C}$ and $\mathbb{C}[q^{\pm 1}]$-algebra homomorphism

$$\widetilde{\phi}_{q=1} : H_{\mathbb{C}[q^{\pm 1}]} \to B_{r,s}(N), \quad E_{r,r+1} \mapsto E_{r,r+1},\ S_i \mapsto (i,i+1),$$

which can be justified by checking the defining relations. For example, we have

$$\widetilde{\phi}_{q=1}(E_{r,r+1}^2 - \delta E_{r,r+1}) = E_{r,r+1}^2 - \phi_{q=1}\Big(\frac{q^N - q^{-N}}{q - q^{-1}}\Big) E_{r,r+1} = E_{r,r+1}^2 - N E_{r,r+1} = 0.$$

Note that $\mathcal{L}_k \in H_{\mathbb{C}[q^{\pm 1}]}$ and $\phi_{q=1}(\mathcal{L}_k) = L_k$ for all $1 \leq k \leq r+s$. Hence we have

$$0 = \widetilde{\phi}_{q=1}\Big(\sum_{i=1}^{d} a_i(q) P_i(\mathcal{L}_1,\dots,\mathcal{L}_{r+s})\Big) = \sum_{i=1}^{d} a_i(1) P_i\big(L_1,\dots,L_{r+s}\big).$$

It follows that $a_i(1) = 0$ for all $1 \le i \le d$, which yields a contradiction since $a_{i_0}(1) \neq 0$. Thus $\{P_i(\mathcal{L}_1, \ldots, \mathcal{L}_{r+s}) \in H_{r,s}(N) \mid 1 \le i \le d\}$ is linearly independent over $\mathbb{C}(q)$, as desired.

Next, we consider the set

$$\{P_i(\mathcal{L}_1, \ldots, \mathcal{L}_{r+s}) \in H_{r,s}(q,\rho) \mid 1 \le i \le d\} \tag{6.3}$$

and assume that there exists a nontrivial $\mathbb{C}(q,\rho)$-linear relation

$$\sum_{i=1}^{d} a_i(q,\rho) P_i(\mathcal{L}_1, \ldots, \mathcal{L}_{r+s}) = 0. \tag{6.4}$$

By multiplying the least common multiple of the denominators and then dividing a suitable power of $\rho - q^N$, we may assume that $a_i(q,\rho) \in \mathbb{C}[q,\rho]$ $(1 \le i \le d)$ and there exists i_0 such that $a_{i_0}(q, q^N) \neq 0$. Now consider the ring homomorphism $\phi_{\rho=q^N} : \mathbf{A} \to \mathbb{C}(q)$ given by $q \mapsto q,\ \rho \mapsto q^N$ and the $\mathbf{A}$-algebra homomorphism given by

$$\widetilde{\phi}_{\rho=q^N} : H_{\mathbf{A}} \to H_{r,s}(N) \quad E_{r,r+1} \mapsto E_{r,r+1},\ S_i \mapsto S_i,$$

where $\mathbf{A} = \mathbb{C}[q^{\pm 1}, \rho^{\pm 1}, \frac{\rho - \rho^{-1}}{q - q^{-1}}] \subset \mathbb{C}(q,\rho)$ and $H_{\mathbf{A}}$ is the $\mathbf{A}$-subalgebra of $H_{r,s}(q,\rho)$ generated by $E_{r,r+1}$ and S_i's. Since $\mathcal{L}_k \in H_{\mathbf{A}}$ and $\phi_{q=\rho^N}(\mathcal{L}_k) = \mathcal{L}_k$ for all $1 \le k \le r+s$, applying $\widetilde{\phi}_{\rho=q^N}$ to (6.4) yields a contradiction to the facts that $a_{i_0}(q, q^N) \neq 0$, similarly as before. Hence the set (6.3) is linearly independent over $\mathbb{C}(q,\rho)$, as desired. $\square$

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